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Plateau-compatible relational scalars: a second-order classical intertwiner

Landmark VII-A

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Abstract

We consider a minimally coupled scalar field on a formal metric $\mathbf{g} = \eta + \kappa h + \kappa^2 k$ and specify a compactly switched scalar action together with unswitched harmonic coordinates having fixed Cauchy data. The admitted gauge transformations are supported strictly inside a region on which the interaction switch equals one and are the identity near the initial surface. For this restricted group we prove an exact first-order cancellation and a second-order classical intertwining identity. All switch derivatives are retained.

The proof uses the covariance of the switched, density-valued Euler operator under plateau-supported transformations and uniqueness of the coefficientwise Cauchy problems. We give the complete second-order relational coefficient, verify its transformation directly, and show that the construction restricts to a precisely defined plateau Einstein–scalar coupling jet. We also derive the first-order defect for larger gauge supports and explain why it cannot be repaired by adding a higher-order coefficient without changing the first-order construction.

A causal support bound and a classical comparison map between switches are proved. Neither equality for switches sharing only an observation plateau nor a quantum algebraic cutoff comparison is asserted. Renormalized products, quantum Ward identities, quantum responses, and counterterm separation are not constructed here. The

result closes the classical gate of the finite-time programme, not its radiative gates.

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Scope, conventions, and the claim being proved

The finite-time choice

This paper makes one choice throughout: harmonic coordinates are defined by a finite-time Cauchy problem, not by a Feynman inverse. The interaction switch is compactly supported, while the harmonic-coordinate equation is unswitched. Compatibility is obtained by restricting the admitted gauge transformations to a declared interaction plateau.

The effective-field-theory motivation is distinct from a claim of ultraviolet completion. Perturbative, locally covariant treatments of gravity provide an appropriate broader setting [2]. The present paper proves only a

classical compatibility statement needed before attempting quantum products.

The principal result is the following, stated informally here and precisely in Theorem : For the switched scalar action specified below, the scalar solution with fixed free Cauchy data and the unswitched harmonic-coordinate map transform by the same pullback under plateau-supported formal gauge transformations. Their relational composition is therefore invariant through second order. The proof is off shell in the metric coefficients. Consequently it remains valid on any invariant restriction of those coefficients, including the Einstein–scalar jet defined in Section .

What is and is not inherited from an earlier obstruction

The motivating obstruction concerns a particular combination of unswitched coordinates, switched scalar response, fixed free input, and gauge parameters extending outside the interaction plateau. Restriction of the gauge domain does not refute that obstruction.

For self-containment, no numbered result from another Landmark manuscript is required here. Section derives the relevant first-order defect and a concrete large-support witness directly. Thus the restriction adopted below is compatible with the large-group obstruction referred to in the production plan as Landmark VI, Theorem 12, without requiring its unavailable statement as a premise.

Necessary specifications

Several choices are mathematically essential.

1. A first-order response does not determine a second-order action. We use a single switch multiplying the entire difference between the curved and flat scalar Lagrangian densities.

2. The metric is a genuine second-order jet

$$\mathbf{g} = \eta + \kappa h + \kappa^2 k + \mathcal{O}(\kappa^3).$$

There is no factor of 1/2 in the definition of k.

3. The scalar Euler operator is density valued. Its infinitesimal transformation consequently contains $\partial_\mu \xi^\mu$.

4. Compactly supported vector fields form a Lie algebra. The admitted group is generated by their flows; its second-order formal jets include both a first-order and a second-order gauge parameter.

5. Every inverse used below has prescribed Cauchy data. No identity of the form EP = id is used on functions with unspecified initial data.

Non-goals

No global removal of the interaction switch is proved. No invariance under the full compactly supported diffeomorphism group is claimed for a fixed switch. We do not construct a quantum gravitational observable algebra, a quantum constraint algebra, a renormalized response, a counterterm catalogue, or an ultraviolet completion.

In particular, the classical result below is not itself a radiative prediction.

Cauchy problems and the admitted gauge domain

Spacetime and free propagation

Let

$$M = \mathbb{R}^{1,3}, \qquad \eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1), \qquad \Sigma_0 = \{t = t_0\},$$

and write

$$M_+ = \{t \geq t_0\}, \qquad M_+^\circ = \{t > t_0\}.$$

All coefficient fields are smooth up to Σ_0. Indices are raised and lowered with η unless stated otherwise.

For $m \geq 0$ set

$$P_m = \mathrm{Box}_\eta - m^2, \qquad \mathrm{Box}_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu.$$

The operator E_m denotes the zero-Cauchy-data solution operator:

$$P_m \mathrm{E}_m F = F, \qquad (\mathrm{E}_m F)|_{\Sigma_0} = 0, \qquad \partial_t (\mathrm{E}_m F)|_{\Sigma_0} = 0.$$

For sources supported away from Σ_0, it agrees on M_+ with the ordinary retarded Green operator applied to their extension by zero.

We use E_m on smooth sources. Sources need not have globally compact spatial support: on every compact observation region, finite propagation reduces the Cauchy problem to a compact portion of the source and initial data. The relevant existence, uniqueness, and finite-propagation statements are standard Cauchy theory for normally hyperbolic operators; see [1, Chapter 3].

Lemma (The left-inverse identity with its data condition). *If $\psi \in C^\infty(M_+)$ satisfies*

$$\psi|_{\Sigma_0} = 0, \qquad \partial_t \psi|_{\Sigma_0} = 0,$$

then

$$\mathrm{E}_m P_m \psi = \psi.$$

More generally,

$$\mathrm{E}_m P_m \psi = \psi - H_m[\psi],$$

where $H_m[\psi]$ is the homogeneous solution with the same Cauchy data as ψ.

Proof. The difference $\psi - \mathrm{E}_m P_m \psi$ solves the homogeneous equation and has the same Cauchy data as ψ. Uniqueness gives ; the zero-data case gives .

Fix a smooth free scalar

$$P_m \varphi_0 = 0.$$

Its Cauchy data will remain fixed under all comparisons and admitted gauge transformations below.

Switch and plateau

Fix

$$g \in C_c^\infty(M_+^\circ), \qquad 0 \le g \le 1, \qquad P_g := \operatorname{int}\{x : g(x) = 1\} \neq \varnothing.$$

Choose a nonempty relatively compact open set U such that

$$\overline{U} \subset P_g.$$

In particular, $\overline{U}$ is separated from Σ_0.

Let

$$\mathfrak{g}_U = C_c^\infty(U; TM).$$

It is closed under the ordinary Lie bracket.

Definition (Admitted group and formal jets). The group $\mathscr{G}_U$ is the subgroup of orientation-preserving diffeomorphisms of M generated by flows of vector fields in $\mathfrak{g}_U$. Its elements are the identity outside a compact subset of U and near Σ_0.

At second perturbative order we use pullbacks of the form

$$F_\kappa^* = \exp\left(\kappa \mathscr{L}_\xi + \kappa^2 \mathscr{L}_\zeta\right) \bmod \kappa^3, \qquad \xi, \zeta \in \mathfrak{g}_U.$$

The Lie algebra of formal transformations is closed modulo κ^3 by the Baker–Campbell–Hausdorff formula. In particular, omitting ζ would not give a convenient composition-stable parametrization.

For scalars define

$$X_\xi = \xi^\mu \partial_\mu.$$

For scalar densities, represented relative to d^4x, define

$$D_\xi = X_\xi + \partial_\mu \xi^\mu.$$

Thus

$$F_\kappa^*\psi = \psi + \kappa X_\xi \psi + \kappa^2\left(X_\zeta + \tfrac{1}{2}X_\xi^2\right)\psi + \mathcal{O}(\kappa^3),$$

$$F_{\kappa,\mathrm{dens}}^* r = r + \kappa D_\xi r + \kappa^2\left(D_\zeta + \tfrac{1}{2}D_\xi^2\right)r + \mathcal{O}(\kappa^3).$$

Every $F \in \mathcal{G}_U$ preserves g:

$$F^* g = g.$$

Indeed, $g = 1$ on U, $F(U) = U$, and F is the identity elsewhere.

The specified switched scalar action

Formal metric and density coefficients

Work coefficientwise in

$$C^\infty(M_+)[[\kappa]]/(\kappa^3).$$

No convergence of the formal series or global invertibility of a perturbed coordinate system is assumed.

Let

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \kappa^2 k_{\mu\nu} + \mathcal{O}(\kappa^3),$$

where h and k are smooth symmetric tensors. Put

$$\tau(h) = \eta^{\mu\nu} h_{\mu\nu}, \qquad h^2 = h_{\mu\nu} h^{\mu\nu},$$

and define

$$a(h) = \tfrac{1}{2}\tau(h), \qquad b(h) = \tfrac{1}{8}\tau(h)^2 - \tfrac{1}{4}h^2,$$

$$A^{\mu\nu}(h) = \tfrac{1}{2}\tau(h)\eta^{\mu\nu} - h^{\mu\nu},$$

$$B^{\mu\nu}(h) = h^\mu{}_\rho h^{\rho\nu} - \tfrac{1}{2}\tau(h)h^{\mu\nu} + b(h)\eta^{\mu\nu}.$$

Writing

$$\rho_{\mathbf{g}} = \sqrt{-\det \mathbf{g}}, \qquad \mathrm{A}^{\mu\nu}_{\mathbf{g}} = \rho_{\mathbf{g}} \mathbf{g}^{\mu\nu},$$

one obtains

$$\rho_{\mathbf{g}} = 1 + \kappa a(h) + \kappa^2 (a(k) + b(h)) + \mathscr{O}(\kappa^3),$$

$$\mathrm{A}^{\mu\nu}_{\mathbf{g}} = \eta^{\mu\nu} + \kappa A^{\mu\nu}(h) + \kappa^2 (A^{\mu\nu}(k) + B^{\mu\nu}(h)) + \mathscr{O}(\kappa^3).$$

Lemma (Density expansion). *Equations – follow from* .

Proof. The inverse metric is

$$\mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 (h^{\mu}{}_{\rho} h^{\rho\nu} - k^{\mu\nu}) + \mathscr{O}(\kappa^3).$$

Expanding $\exp\left(\frac{1}{2}\mathrm{tr}\log\left(\mathrm{id} + \kappa\eta^{-1}h + \kappa^2\eta^{-1}k\right)\right)$ gives . Multiplication gives .

Action-level switching

Define

$$\mathrm{A}_g[\mathbf{g}]^{\mu\nu} = \eta^{\mu\nu} + g(\mathrm{A}^{\mu\nu}_{\mathbf{g}} - \eta^{\mu\nu}),$$

$$\rho_g[\mathbf{g}] = 1 + g(\rho_{\mathbf{g}} - 1).$$

The switched scalar action is

$$S^{\mathrm{sc}}_g[\mathbf{g}, \varphi] = -\frac{1}{2} \int_{M_+} \left(\mathrm{A}_g[\mathbf{g}]^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi + m^2 \rho_g[\mathbf{g}] \varphi^2 \right) \mathrm{d}^4 x.$$

The action is understood variationally, using compactly supported variations or variations respecting the prescribed boundary data; its numerical value need not be finite on every smooth configuration.

This prescription is equivalent to multiplying the difference between the curved and flat scalar Lagrangian densities by g. In particular, the quadratic metric vertex carries one factor of g, not a factor of g^2. Two first-order insertions will nevertheless produce two switch factors in the iterated response.

The corresponding Euler operator is

$$\mathrm{D}_g[\mathbf{g}]\varphi = \partial_\mu\big(\mathrm{A}_g[\mathbf{g}]^{\mu\nu}\partial_\nu\varphi\big) - m^2\rho_g[\mathbf{g}]\varphi.$$

Its output represents a scalar density relative to d^4x.

Coefficient operators and explicit switch derivatives

Define

$$\mathrm{Q}_h^{(m)}\psi = \partial_\mu\big(A^{\mu\nu}(h)\partial_\nu\psi\big) - m^2 a(h)\psi,$$

$$\mathrm{T}_h^{(m)}\psi = \partial_\mu\big(B^{\mu\nu}(h)\partial_\nu\psi\big) - m^2 b(h)\psi.$$

Then

$$\mathrm{D}_g[\mathbf{g}] = P_m + \kappa\mathrm{V}_g[h] + \kappa^2\mathrm{W}_g[h,k] + \mathscr{O}(\kappa^3),$$

where

$$\mathrm{V}_g[h]\psi = \partial_\mu\big(gA^{\mu\nu}(h)\partial_\nu\psi\big) - m^2 g\, a(h)\psi = \mathrm{Q}_{gh}^{(m)}\psi,$$

$$\mathrm{W}_g[h,k]\psi = \partial_\mu\big(g(A^{\mu\nu}(k) + B^{\mu\nu}(h))\partial_\nu\psi\big) - m^2 g(a(k) + b(h))\psi.$$

In expanded form,

$$\mathrm{V}_g[h]\psi = g\mathrm{Q}_h^{(m)}\psi + (\partial_\mu g)A^{\mu\nu}(h)\partial_\nu\psi,$$

$$\mathrm{W}_g[h,k]\psi = g(\mathrm{Q}_k^{(m)} + \mathrm{T}_h^{(m)})\psi$$

$$+ (\partial_\mu g)(A^{\mu\nu}(k) + B^{\mu\nu}(h))\partial_\nu\psi.$$

These identities display all derivatives of the switch in the coefficient operators. In particular, $\mathrm{V}_g[h] \neq g\mathrm{Q}_h^{(m)}$ in general.

Remark (Switching is a choice of model). The prescription is part of the theorem, not a consequence of the first-order repair. A different assignment of switch factors to higher vertices must be declared separately. The

proof below applies to the action actually written here.

The switched solution with fixed free input

Define

$$\varphi_g[\mathbf{g};\varphi_0] = \varphi_0 + \kappa\varphi_{1,g} + \kappa^2\varphi_{2,g} + \mathscr{O}(\kappa^3)$$

by

$$\mathrm{D}_g[\mathbf{g}]\varphi_g = 0 \bmod \kappa^3$$

and by requiring all positive-order scalar coefficients to have zero Cauchy data. Coefficient comparison gives

$$\begin{aligned}\varphi_{1,g} &= -\mathrm{E}_m \mathrm{V}_g[h]\varphi_0,\\ \varphi_{2,g} &= -\mathrm{E}_m(\mathrm{V}_g[h]\varphi_{1,g} + \mathrm{W}_g[h,k]\varphi_0)\\ &= \mathrm{E}_m \mathrm{V}_g[h]\mathrm{E}_m \mathrm{V}_g[h]\varphi_0 - \mathrm{E}_m \mathrm{W}_g[h,k]\varphi_0.\end{aligned}$$

Each source to which E_m is applied in these formulas is smooth and compactly supported, because the coefficients of V_g and W_g are compactly supported. Thus these formulas define smooth classical coefficients without any product of singular distributions.

Lemma (Uniqueness of the switched scalar jet). *Equations – give the unique scalar jet satisfying the switched equation and the specified Cauchy data.*

Proof. At order κ the equation is

$$P_m\varphi_{1,g} = -\mathrm{V}_g[h]\varphi_0.$$

At order κ^2 it is

$$P_m\varphi_{2,g} = -\mathrm{V}_g[h]\varphi_{1,g} - \mathrm{W}_g[h,k]\varphi_0.$$

Existence follows from the Cauchy solution operator, and uniqueness follows successively at the two orders.

Unswitched harmonic coordinates

Definition and boundary prescription

The four functions $\mathrm{X}^{\mu}_{\mathbf{g}}$ are defined coefficientwise by

$$\partial_{\alpha}\big(\mathrm{A}^{\alpha\beta}_{\mathbf{g}}\partial_{\beta}\mathrm{X}^{\mu}_{\mathbf{g}}\big) = 0,$$

with Cauchy data

$$\mathrm{X}^{\mu}_{\mathbf{g}}|_{\Sigma_0} = x^{\mu}|_{\Sigma_0}, \qquad \partial_t \mathrm{X}^{\mu}_{\mathbf{g}}|_{\Sigma_0} = \partial_t x^{\mu}.$$

There is no switch in .

Write

$$\mathrm{X}^{\mu}_{\mathbf{g}} = x^{\mu} + \kappa w^{\mu}[h] + \kappa^2 v^{\mu}[h,k] + \mathscr{O}(\kappa^3).$$

Define

$$C^{\mu}[h] = \partial_{\nu} h^{\nu\mu} - \frac{1}{2}\partial^{\mu}\tau(h) = -\partial_{\nu} A^{\nu\mu}(h).$$

Then

$$w^{\mu}[h] = \mathrm{E}_0 C^{\mu}[h],$$

$$v^{\mu}[h,k] = \mathrm{E}_0 C^{\mu}[k] - \mathrm{E}_0\big(\partial_{\alpha} B^{\alpha\mu}(h) + \mathrm{Q}^{(0)}_{h} w^{\mu}[h]\big).$$

Lemma (Existence and uniqueness of the coordinate jet). *Equations – uniquely solve – through second order.*

Proof. At first order,

$$\mathrm{Box}_{\eta}\, w^{\mu} = -\partial_{\alpha} A^{\alpha\mu}(h) = C^{\mu}[h].$$

At second order,

$$\mathrm{Box}_{\eta}\, v^{\mu} = C^{\mu}[k] - \partial_{\alpha} B^{\alpha\mu}(h) - \mathrm{Q}^{(0)}_{h} w^{\mu}.$$

Both coefficients have zero Cauchy data. Existence and uniqueness follow as in Lemma .

The sources in – need not be compactly supported. Their use is legitimate as smooth Cauchy sources; the causal support bound proved later makes the dependence on a compact observation region explicit.

Formal inversion

A map with leading term id has a unique formal inverse. In the present notation,

$$(\mathrm{X}_{\mathbf{g}}^{-1})^{\mu}(y) = y^{\mu} - \kappa w^{\mu}(y) + \kappa^{2}\big(w^{\nu}\partial_{\nu}w^{\mu} - v^{\mu}\big)(y) + \mathcal{O}(\kappa^{3}).$$

This is a formal statement. It does not assert that a nonperturbative harmonic-coordinate map is a global diffeomorphism.

Proof. Substitution into $\mathrm{X}_{\mathbf{g}}^{\mu}(x) = x^{\mu} + \kappa w^{\mu}(x) + \kappa^{2}v^{\mu}(x)$ cancels the first-order terms. At second order, Taylor expansion contributes $-w^{\nu}\partial_{\nu}w^{\mu}$, which is cancelled by the displayed inverse coefficient.

The relational scalar

For a fixed label-space test density represented by $f \in C_{c}^{\infty}(U)$, define

$$\Phi_{g}[\mathbf{g};\varphi_{0}](y) = \varphi_{g}[\mathbf{g};\varphi_{0}]\big(\mathrm{X}_{\mathbf{g}}^{-1}(y)\big),$$

$$\widetilde{\mathcal{O}}_{f}[g](\mathbf{g},\varphi_{0}) = \int f(y)\Phi_{g}[\mathbf{g};\varphi_{0}](y)\,\mathrm{d}^{4}y.$$

The smearing is performed in coordinate-label space. Therefore no spacetime volume density is to be inserted into . An equivalent spacetime expression would contain the Jacobian of $\mathrm{X}_{\mathbf{g}}$.

Expanding,

$$\Phi_{g} = \varphi_{0} + \kappa A_{1}^{(g)} + \kappa^{2}A_{2}^{(g)} + \mathcal{O}(\kappa^{3}),$$

where

$$A_{1}^{(g)} = \varphi_{1,g} - w^{\mu}\partial_{\mu}\varphi_{0},$$

$$A_2^{(g)} = \varphi_{2,g} - w^\mu \partial_\mu \varphi_{1,g} + \left(w^\nu \partial_\nu w^\mu - v^\mu \right) \partial_\mu \varphi_0$$

$$+ \frac{1}{2} w^\mu w^\nu \partial_\mu \partial_\nu \varphi_0.$$

The notation $A_1^{(g)}$ emphasizes that the first-order expression itself is unchanged; its admitted gauge domain is changed.

Exact first-order repair and the larger-support defect

Two linear identities

Define

$$(K\xi)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = (\mathscr{L}_\xi \eta)_{\mu\nu}.$$

Lemma (Coordinate identity). *For every smooth vector field ξ,*

$$C^\mu[K\xi] = \mathrm{Box}_\eta \, \xi^\mu.$$

If ξ vanishes near Σ_0, then

$$w[K\xi] = \xi.$$

Proof. The trace of $K\xi$ is $2\partial_\rho \xi^\rho$, so

$$\partial_\nu (K\xi)^{\nu\mu} = \mathrm{Box}_\eta \, \xi^\mu + \partial^\mu \partial_\rho \xi^\rho.$$

The second term is cancelled by half the derivative of the trace. Equation then follows from Lemma , componentwise.

Lemma (Density-valued scalar identity). *For every smooth scalar ψ,*

$$\mathrm{Q}^{(m)}_{K\xi} \psi = D_\xi P_m \psi - P_m X_\xi \psi.$$

In particular, if $P_m \varphi_0 = 0$, then

$$\mathrm{Q}^{(m)}_{K\xi}\varphi_0 = -P_m(X_\xi\varphi_0).$$

Proof. The unswitched density-valued operator

$$\mathrm{D}[\mathbf{g}]\psi = \partial_\mu\big(\rho_{\mathbf{g}}\mathbf{g}^{\mu\nu}\partial_\nu\psi\big) - m^2\rho_{\mathbf{g}}\psi$$

satisfies the change-of-variables identity

$$\mathrm{D}[F^*\mathbf{g}](F^*\psi) = F^*_{\mathrm{dens}}(\mathrm{D}[\mathbf{g}]\psi).$$

Differentiate at $\mathbf{g} = \eta$ along the flow of ξ. The variation of the operator is $\mathrm{Q}^{(m)}_{K\xi}$, the variation of the scalar is $X_\xi\psi$, and the variation of the output density is $D_\xi P_m\psi$. Rearrangement gives .

The plateau theorem

Theorem (Exact first-order plateau repair). *Let the data satisfy Section . For every smooth metric coefficient h and every $\xi \in \mathfrak{g}_U$,*

$$A_1^{(g)}[h + K\xi; \varphi_0] = A_1^{(g)}[h; \varphi_0].$$

More explicitly,

$$gK\xi = K\xi, \qquad \varphi_{1,g}[h + K\xi] - \varphi_{1,g}[h] = X_\xi\varphi_0, \qquad w[h + K\xi] - w[h] = \xi.$$

Consequently, the first-order coefficient of $\widehat{\mathscr{O}}_f[g]$ is invariant under the infinitesimal transformations admitted by $\mathscr{G}_U$.

Proof. All derivatives of ξ are supported in U, where $g = 1$. Hence $gK\xi = K\xi$. By linearity and Lemma ,

$$\begin{aligned}\varphi_{1,g}[h + K\xi] - \varphi_{1,g}[h] &= -\mathrm{E}_m\mathrm{Q}^{(m)}_{gK\xi}\varphi_0 \\ &= \mathrm{E}_m P_m(X_\xi\varphi_0) \\ &= X_\xi\varphi_0.\end{aligned}$$

The last equality is valid because ξ vanishes near Σ_0. Lemma gives the coordinate variation. Substitution into

proves .

The theorem is stronger than a statement restricted to $h = 0$ or $h = K\xi'$: no field equation for h is required.

Limits of the first-order statement. Theorem alone proves neither second-order covariance, independence of (g, U), independence of the dressing prescription, a full gravitational gauge quotient, nor a quantum Ward identity. Second-order covariance is established separately below; the other claims are not made.

The defect outside the plateau

Proposition (Exact first-order defect). *Let* $\xi \in C_c^\infty(M_+^\circ; TM)$*, without requiring* $\operatorname{supp}\xi \subset U$*. Then*

$$A_1^{(g)}[h + K\xi] - A_1^{(g)}[h] = \mathrm{E}_m \mathrm{Q}^{(m)}_{(1-g)K\xi}\varphi_0.$$

Proof. The coordinate variation is still ξ, since the initial-data condition is unchanged. Thus the left-hand side is

$$-\mathrm{E}_m \mathrm{Q}^{(m)}_{gK\xi}\varphi_0 - X_\xi\varphi_0.$$

By and Lemma ,

$$X_\xi\varphi_0 = -\mathrm{E}_m \mathrm{Q}^{(m)}_{K\xi}\varphi_0.$$

Linearity of Q in its tensor argument gives .

Proposition (A large-support witness). *There are compactly supported gauge parameters after* Σ_0 *for which is nonzero, including parameters that are constant on a neighbourhood of* $\operatorname{supp} g$*.*

Proof. Choose $\chi \in C_c^\infty(M_+^\circ)$ with $\chi = 1$ on a neighbourhood of $\operatorname{supp} g$, and let

$$\xi^\mu = c^\mu\chi$$

for a nonzero constant vector c. Then $K\xi = 0$ on a neighbourhood of $\operatorname{supp} g$, hence

$$gK\xi = 0.$$

The switched scalar correction has zero variation, while the coordinate correction changes by ξ. Therefore

$$A_1^{(g)}[h + K\xi] - A_1^{(g)}[h] = -X_\xi \varphi_0.$$

Choose free Cauchy data such that $c^\mu \partial_\mu \varphi_0$ is nonzero at some point of U. Such data exist, for example by taking an appropriate real Klein–Gordon plane wave and restricting its smooth Cauchy data. A test function supported sufficiently close to that point gives a nonzero smeared variation.

This ξ is not supported in U: its nontrivial cutoff to zero takes place outside the region on which it is constant, and that region contains $\operatorname{supp} g$.

Corollary (A higher coefficient cannot repair this defect). *With fixed free input and the same first-order coefficient $A_1^{(g)}$, adding a regular coefficient $\kappa^2 B_2$ cannot make the observable invariant under every gauge parameter of Proposition .*

Proof. In a formal series with nonnegative powers of κ, invariance requires the coefficient at each order to vanish. A nonzero coefficient at order κ cannot be cancelled by a coefficient at order κ^2.

Plateau covariance and the second-order intertwiner

The metric transformation law

Let F_κ be as in . The transformed metric jet is

$$\mathbf{g}' = F_\kappa^* \mathbf{g} = \eta + \kappa h' + \kappa^2 k' + \mathcal{O}(\kappa^3),$$

with

$$h' = h + K\xi,$$

$$k' = k + \mathscr{L}_\xi h + K\zeta + \frac{1}{2}\mathscr{L}_\xi^2 \eta.$$

The second equation is indispensable. Keeping k fixed while changing h by $K\xi$ does not describe the second-order pullback of a metric.

Covariance of the switched density operator

Lemma (Plateau covariance). *For $F \in \mathcal{G}_U$ and every smooth scalar ψ,*

$$\mathrm{D}_g[F^*\mathbf{g}](F^*\psi) = F^*_{\mathrm{dens}}(\mathrm{D}_g[\mathbf{g}]\psi).$$

The identity also holds coefficientwise for the formal transformations .

Proof. Regard $\mathrm{A}_{\mathbf{g}}$ as a contravariant tensor density and $\rho_{\mathbf{g}}$ as a scalar density. Their unswitched definitions are natural under pullback.

By ,

$$\mathrm{A}_g[\mathbf{g}] = (1 - g)\mathrm{A}_\eta + g\mathrm{A}_{\mathbf{g}}.$$

Since $F^*g = g$,

$$F^*\mathrm{A}_g[\mathbf{g}] = (1 - g)F^*\mathrm{A}_\eta + g\mathrm{A}_{F^*\mathbf{g}}.$$

The difference between this expression and $\mathrm{A}_g[F^*\mathbf{g}]$ is

$$(1 - g)(F^*\mathrm{A}_\eta - \mathrm{A}_\eta).$$

It vanishes: on U the factor $1 - g$ is zero, while outside U the diffeomorphism and its derivative are the identity. Hence

$$F^*\mathrm{A}_g[\mathbf{g}] = \mathrm{A}_g[F^*\mathbf{g}].$$

The same argument applies to $\rho_g[\mathbf{g}]$.

The divergence of a vector density and the pairing with a scalar derivative are natural. Applying these operations proves . Its formal coefficient identities follow by expansion.

Remark (Why switch derivatives have not disappeared). Lemma is an identity for the complete divergence-form operator. It does not replace $\partial_\mu(gA^{\mu\nu}\partial_\nu)$ by $g\partial_\mu(A^{\mu\nu}\partial_\nu)$. The terms in and remain part of the

operator.

Scalar and coordinate intertwiners

Proposition (Cauchy intertwiners). *For the transformed jet –,*

$$\varphi_g[F_\kappa^*\mathbf{g};\varphi_0] = F_\kappa^*\varphi_g[\mathbf{g};\varphi_0] \bmod \kappa^3,$$

$$\mathrm{X}_{F_\kappa^*\mathbf{g}} = \mathrm{X}_{\mathbf{g}} \circ F_\kappa \bmod \kappa^3.$$

The free input φ_0 is the same on both sides.

Proof. By Lemma , the right-hand side of solves the transformed switched equation. Since F_κ is the identity near Σ_0, its Cauchy data are exactly those of φ_0. Uniqueness in Lemma proves the first identity.

For the second identity, each component of $\mathrm{X}_{\mathbf{g}}$ is a scalar solution of the unswitched, massless density equation. Ordinary naturality of that equation implies that $\mathrm{X}_{\mathbf{g}} \circ F_\kappa$ solves the harmonic equation for $F_\kappa^*\mathbf{g}$. Its Cauchy data agree with , again because F_κ is the identity near Σ_0. Lemma gives the result.

Coefficient extraction yields

$$\varphi'_{1,g} = \varphi_{1,g} + X_\xi\varphi_0,$$

$$\varphi'_{2,g} = \varphi_{2,g} + X_\xi\varphi_{1,g} + X_\zeta\varphi_0 + \frac{1}{2}X_\xi^2\varphi_0,$$

$$w'^\mu = w^\mu + \xi^\mu,$$

$$v'^\mu = v^\mu + \xi^\nu\partial_\nu w^\mu + \zeta^\mu + \frac{1}{2}\xi^\nu\partial_\nu\xi^\mu.$$

Theorem (Second-order plateau-compatible relational scalar). *For the switched action , the harmonic Cauchy prescription –, and the formal gauge transformations admitted by $\mathscr{G}_U$,*

$$\Phi_g[F_\kappa^*\mathbf{g};\varphi_0] = \Phi_g[\mathbf{g};\varphi_0] \bmod \kappa^3.$$

Consequently,

$$\widehat{\mathscr{O}}_f[g](F_\kappa^*\mathbf{g}, \varphi_0) = \widehat{\mathscr{O}}_f[g](\mathbf{g}, \varphi_0) \bmod \kappa^3.$$

The coefficients are and , with , , , and .

Proof. Formal inversion of gives

$$\mathrm{X}_{F_\kappa^*\mathbf{g}}^{-1} = F_\kappa^{-1} \circ \mathrm{X}_{\mathbf{g}}^{-1} \bmod \kappa^3.$$

Together with ,

$$\Phi_g[F_\kappa^*\mathbf{g}; \varphi_0] = (\varphi_g[\mathbf{g}; \varphi_0] \circ F_\kappa) \circ (F_\kappa^{-1} \circ \mathrm{X}_{\mathbf{g}}^{-1})$$

$$= \varphi_g[\mathbf{g}; \varphi_0] \circ \mathrm{X}_{\mathbf{g}}^{-1} \bmod \kappa^3.$$

This proves . Smearing proves . Taylor expansion gives the stated coefficients.

The pointwise identity is valid wherever the formal coefficients are evaluated. The restriction $\operatorname{supp} f \subset U$ is the observation contract used in this paper; it is not needed for the algebraic cancellation itself.

An operator-level second-order audit

It is useful to verify that the second-order assertion contains the correct density terms. Write

$$\mathrm{V} = \mathrm{V}_g[h], \qquad \mathrm{W} = \mathrm{W}_g[h, k], \qquad \mathrm{V}' = \mathrm{V}_g[h'], \qquad \mathrm{W}' = \mathrm{W}_g[h', k'].$$

Set

$$X = X_\xi, \qquad T_2 = X_\zeta + \tfrac{1}{2}X_\xi^2, \qquad D = D_\xi, \qquad \Delta_2 = D_\zeta + \tfrac{1}{2}D_\xi^2.$$

Expanding Lemma gives

$$\mathrm{V}' = \mathrm{V} + DP_m - P_m X,$$

$$\mathrm{W}' + \mathrm{V}'X + P_m T_2 = \mathrm{W} + D\mathrm{V} + \Delta_2 P_m.$$

These are identities of differential operators, not only identities on free solutions. In particular, they retain the terms proportional to P_m.

To verify the second scalar transformation directly, put

$$a = \varphi_{1,g}, \qquad b = \varphi_{2,g}, \qquad b_{\mathrm{cand}} = b + Xa + T_2\varphi_0.$$

Since $P_m a = -\mathrm{V}\varphi_0$ and $P_m b = -\mathrm{V}a - \mathrm{W}\varphi_0$, equations – imply

$$\begin{aligned} P_m b_{\mathrm{cand}} &= -\mathrm{V}a - \mathrm{W}\varphi_0 + P_m X a + P_m T_2 \varphi_0 \\ &= -\mathrm{V}a - \mathrm{W}\varphi_0 - D\mathrm{V}\varphi_0 + (\mathrm{V} - \mathrm{V}')a \\ &\quad + \mathrm{W}\varphi_0 + D\mathrm{V}\varphi_0 - \mathrm{W}'\varphi_0 - \mathrm{V}' X\varphi_0 \\ &= -\mathrm{V}'(a + X\varphi_0) - \mathrm{W}'\varphi_0. \end{aligned}$$

The candidate has zero Cauchy data. Uniqueness proves independently of any informal cancellation of switch terms.

The full second-order coefficient

Substituting the constructed scalar coefficients into gives the explicit expression

$$\begin{aligned} A_2^{(g)} &= \mathrm{E}_m \mathrm{V}_g[h]\mathrm{E}_m \mathrm{V}_g[h]\varphi_0 - \mathrm{E}_m \mathrm{W}_g[h,k]\varphi_0 \\ &\quad + w^\mu \partial_\mu \big(\mathrm{E}_m \mathrm{V}_g[h]\varphi_0\big) + \big(w^\nu \partial_\nu w^\mu - v^\mu\big)\partial_\mu \varphi_0 \\ &\quad + \frac{1}{2} w^\mu w^\nu \partial_\mu \partial_\nu \varphi_0. \end{aligned}$$

There are five displayed contributions: iterated scalar propagation, the second-order scalar vertex, the mixed first-order term, the second-order inverse-coordinate term, and the quadratic Taylor term.

None may be omitted merely because another term is described as a dressing correction.

Restriction to a named Einstein–scalar coupling jet

Coupling expansion versus a scalar background

For definiteness, normalize the unswitched Einstein–scalar action as

$$S_{\mathrm{ES}}[\mathbf{g},\varphi] = \frac{1}{2\kappa^2}\int \rho_{\mathbf{g}} R[\mathbf{g}]\,\mathrm{d}^4x - \frac{1}{2}\int \rho_{\mathbf{g}}\left(\mathbf{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2\right)\mathrm{d}^4x.$$

Its metric equation is

$$G_{\mu\nu}[\mathbf{g}] = \kappa^2 T_{\mu\nu}[\mathbf{g},\varphi].$$

This fixes the normalization of κ for this section.

The expansion is a coupling expansion. A nonzero free field φ_0 is allowed at order κ^0 because its gravitational source enters at order κ^2. It is not being identified with a nonzero background expectation value in a linearization about a different Einstein–scalar background.

Define the Einstein-tensor coefficients by

$$G_{\mu\nu}[\eta + \kappa h + \kappa^2 k] = \kappa G^{(1)}_{\mu\nu}[h] + \kappa^2\left(G^{(1)}_{\mu\nu}[k] + G^{(2)}_{\mu\nu}[h,h]\right) + \mathcal{O}(\kappa^3).$$

An explicit coefficient definition is given in Appendix .

The leading stress tensor is

$$T^{(0)}_{\mu\nu}[\varphi_0] = \partial_\mu\varphi_0\partial_\nu\varphi_0 - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\rho\varphi_0\partial^\rho\varphi_0 + m^2\varphi_0^2\right).$$

Definition (Plateau Einstein–scalar coupling jet). The space $\mathscr{J}^{\beta}_2$ consists of tuples

$$(h, k, \varphi_0, \varphi_{1,g}, \varphi_{2,g})$$

such that:

1. $P_m\varphi_0 = 0$;

2. the scalar coefficients are the zero-data solutions –;

3. the metric coefficients satisfy

$$G^{(1)}[h] = 0,$$

$$G^{(1)}[k] + G^{(2)}[h,h] = T^{(0)}[\varphi_0].$$

No zero-Cauchy-data condition is imposed on h or k.

The metric equations in this definition are unswitched at the retained order; the scalar equation is switched. On P_g the scalar operator equals the unswitched curved operator, so the tuple satisfies the Einstein–scalar equations there through the stated orders.

This definition does not assert a nonlinear completion of the mixed switched system outside the plateau, nor does it construct a switched, gauge-fixed gravitational action. Those are separate requirements for a later quantum model.

Proposition (Stability of the coupling jet). *The transformations – and – preserve $\mathscr{J}^{\beta}_2$ for the formal transformations admitted by $\mathscr{G}_U$.*

Proof. The scalar equations are preserved by Proposition .

For the metric equations, naturality of the Einstein tensor gives

$$G[F_\kappa^* \mathbf{g}] = F_\kappa^* G[\mathbf{g}].$$

At first order,

$$G^{(1)}[h'] = G^{(1)}[h].$$

At second order,

$$G^{(1)}[k'] + G^{(2)}[h', h'] = G^{(1)}[k] + G^{(2)}[h, h] + \mathscr{L}_\xi G^{(1)}[h].$$

The last term vanishes on . The leading scalar coefficient remains φ_0, so the source $T^{(0)}[\varphi_0]$ in the order-κ^2 equation is unchanged. Thus is preserved.

Corollary. *The relational scalar of Theorem is invariant under the formal transformations admitted by $\mathscr{G}_U$ when restricted to $\mathscr{J}^{\beta}_2$.*

Proof. Theorem holds for arbitrary smooth h, k. Proposition shows that its restriction has an invariant domain.

A nonzero-source example

Example (An explicit sourced jet). Take $m = 0$, a nonzero constant covector a_μ, and

$$\varphi_0 = a_\mu x^\mu, \qquad h = 0.$$

Then $P_0\varphi_0 = 0$, and

$$T_{\mu\nu} = a_\mu a_\nu - \frac{1}{2}\eta_{\mu\nu} a_\rho a^\rho$$

is constant. Let $T = \eta^{\mu\nu}T_{\mu\nu}$ and set

$$\sigma(x) = \left(-\frac{1}{4}T_{\alpha\beta} + \frac{1}{12}\eta_{\alpha\beta}T\right)x^\alpha x^\beta, \qquad k_{\mu\nu} = 2\sigma\eta_{\mu\nu}.$$

For a conformal linearized perturbation in four dimensions,

$$G^{(1)}_{\mu\nu}[2\sigma\eta] = -2\partial_\mu\partial_\nu\sigma + 2\eta_{\mu\nu}\mathrm{Box}_\eta\,\sigma.$$

Here

$$\partial_\mu\partial_\nu\sigma = -\frac{1}{2}T_{\mu\nu} + \frac{1}{6}\eta_{\mu\nu}T, \qquad \mathrm{Box}_\eta\,\sigma = \frac{1}{6}T,$$

so $G^{(1)}[k] = T$ as a tensor equation. Furthermore,

$$\varphi_{1,g} = 0, \qquad \varphi_{2,g} = -\mathrm{E}_0\mathrm{Q}^{(0)}_{gk}\varphi_0.$$

This is a nonzero-source member of $\mathscr{J}^\beta_2$.

The polynomial metric coefficient is used formally on finite causal domains. No asymptotic-flatness assertion or global smallness assertion is attached to this example.

Support and comparison of switches

The observation support is not the functional support

For $f \in C^\infty_c(U)$ define

$$\mathscr{D}_f = J_\eta^-(\operatorname{supp} f) \cap M_+.$$

This set is compact: the time coordinate is bounded between t_0 and the largest observation time, and the spatial projection is bounded by the corresponding Minkowski light cones.

Proposition (Coefficientwise causal dependence). *For fixed free input φ_0, the coefficients through order κ^2 of $\widetilde{\mathcal{O}}_f[g]$ depend on h and k only through their values on every neighbourhood of $\mathscr{D}_f$. The switch dependence has the same causal bound.*

In particular, the functional support in the metric variables is contained in $\mathscr{D}_f$, not necessarily in $\operatorname{supp} f$.

Proof. The Cauchy operator E_m has retarded support. Therefore $\varphi_{1,g}(y)$ and $w(y)$ depend only on sources in $J_\eta^-(y) \cap M_+$.

The local differential operators V_g, W_g, $\mathrm{Q}_h^{(0)}$, and C do not enlarge support. For a second-order nested term, an intermediate point z lies in $J_\eta^-(y)$, and every point affecting its first-order coefficient lies in $J_\eta^-(z) \cap M_+$. Transitivity gives

$$J_\eta^-(z) \cap M_+ \subset J_\eta^-(y) \cap M_+.$$

This proves the asserted bound for $\varphi_{2,g}$ and v, and hence for every term in . Taking the union over $y \in \operatorname{supp} f$ gives $\mathscr{D}_f$.

Because $\mathscr{D}_f$ can meet Σ_0, a future quantum extension must also control distributions associated with the finite-time boundary prescription. Compactness of $\mathscr{D}_f$ in the closed half-space is not a substitute for a wavefront-set or boundary-extension argument.

A valid equality criterion

Proposition (Equality under causal agreement of switches). *Fix h, k, φ_0 and the same harmonic Cauchy prescription. If two switches g_1, g_2 agree on an open neighbourhood of $\mathscr{D}_f$, then*

$$\widehat{\mathcal{O}}_f[g_1] = \widehat{\mathcal{O}}_f[g_2] \bmod \kappa^3.$$

Proof. Their first- and second-order scalar operators agree on that neighbourhood, including all derivatives of the switches. Retarded support and the coefficient formulas then give equality of the scalar coefficients on supp f. The harmonic coordinates do not depend on the switch. Equation follows.

Agreement merely on a neighbourhood of supp f is weaker. At first order the difference is explicitly

$$A_1^{(g_2)} - A_1^{(g_1)} = -\mathrm{E}_m \mathrm{Q}^{(m)}_{(g_2 - g_1)h} \varphi_0.$$

The right-hand side samples the causal past. A common observation plateau alone does not force its source or its retarded propagation to vanish.

Classical comparison with fixed Cauchy data

For a fixed metric jet, abbreviate

$$\mathrm{D}_i = \mathrm{D}_{g_i}[\mathbf{g}].$$

The formal zero-data inverse of D_i through second order is

$$\mathrm{G}_i = \mathrm{E}_m - \kappa \mathrm{E}_m \mathrm{V}_i \mathrm{E}_m$$

$$+ \kappa^2 \big(\mathrm{E}_m \mathrm{V}_i \mathrm{E}_m \mathrm{V}_i \mathrm{E}_m - \mathrm{E}_m \mathrm{W}_i \mathrm{E}_m \big) + \mathcal{O}(\kappa^3).$$

Multiplication verifies $\mathrm{D}_i \mathrm{G}_i = \mathrm{id} \bmod \kappa^3$ on smooth sources. All positive-order corrections have zero Cauchy data.

Let Sol_i denote formal scalar solutions of $\mathrm{D}_i \psi = 0 \bmod \kappa^3$ with arbitrary prescribed formal Cauchy data.

Proposition (Classical comparison map). *The map*

$$r_{21}\psi = \psi - \mathrm{G}_2(\mathrm{D}_2 - \mathrm{D}_1)\psi \bmod \kappa^3$$

sends Sol_1 *to* Sol_2*, preserves Cauchy data, and is the unique map with these properties. It satisfies*

$$r_{32} r_{21} = r_{31}, \qquad r_{12} r_{21} = \mathrm{id} \bmod \kappa^3.$$

Proof. For $\psi \in \mathrm{Sol}_1$,

$$\mathrm{D}_2 r_{21}\psi = \mathrm{D}_2\psi - (\mathrm{D}_2 - \mathrm{D}_1)\psi$$

$$= \mathrm{D}_1\psi = 0 \bmod \kappa^3.$$

The correction has zero Cauchy data. Coefficientwise uniqueness therefore characterizes $r_{21}\psi$ as the D_2 solution with the same data as ψ. Composition and inversion follow from the same characterization.

If a gauge transformation is supported inside a common plateau of g_1 and g_2, uniqueness also gives

$$r_{21}[F_\kappa^* \mathbf{g}](F_\kappa^*\psi) = F_\kappa^*(r_{21}[\mathbf{g}]\psi) \bmod \kappa^3.$$

Thus the classical comparison respects the common restricted gauge domain.

What this comparison does not prove

A classical observable B_1 on Sol_1 can be transported to Sol_2 by

$$\alpha_{21}^{\mathrm{cl}} B_1 = B_1 \circ r_{12}.$$

However, this is not generally the assertion that the same evaluation formula defines the transported observable. For example, if

$$B_i(\psi) = \int f(y)\, \psi\big(\mathrm{X}_{\mathbf{g}}^{-1}(y)\big)\mathrm{d}^4 y, \qquad \psi \in \mathrm{Sol}_i,$$

then

$$(\alpha_{21}^{\mathrm{cl}} B_1)(\psi) = \int f(y)\, (r_{12}\psi)\big(\mathrm{X}_{\mathbf{g}}^{-1}(y)\big)\mathrm{d}^4 y,$$

which need not equal $B_2(\psi)$.

This distinction is the classical counterpart of the need to specify an identification between cutoff-dependent descriptions rather than assert equality in an unchanged realization.

In perturbative algebraic quantum field theory, algebraic cutoff comparison is distinct from a direct numerical or

operator removal of the switch [2, Section 4]. The present classical map does not establish the corresponding quantum isomorphism for the nonlocal harmonic insertion. Such a result would require the product domain and localization hypotheses to be proved.

Consistency checks

Flat metric

If $h = k = 0$, then

$$\varphi_{1,g} = \varphi_{2,g} = w = v = 0.$$

Consequently,

$$\Phi_g = \varphi_0 \bmod \kappa^3$$

for every switch. This is the flat-metric normalization.

It is not appropriate to identify switching off scalar interactions with removing the geometric dressing at a nonzero metric. These are different operations in the construction.

A pure-gauge metric jet

Start from the flat metric and take $\zeta = 0$. Its pullback has

$$h = K\xi, \qquad k = \frac{1}{2}\mathscr{L}_\xi^2 \eta.$$

For $\xi \in \mathfrak{g}_U$, the intertwining formulas give

$$\varphi_{1,g} = X_\xi \varphi_0, \ \varphi_{2,g} = \frac{1}{2} X_\xi^2 \varphi_0,$$

$$w^\mu = \xi^\mu, \qquad v^\mu = \frac{1}{2}\xi^\nu \partial_\nu \xi^\mu.$$

Substitution into – yields

$$A_1^{(g)} = A_2^{(g)} = 0.$$

This check uses the second-order metric coefficient in . The straight-line family $\eta + \kappa K\xi$ with $k = 0$ is generally not a pure-gauge family through second order.

The geometric second-order term is necessary

The expression

$$\varphi_{2,g} - X_w \varphi_{1,g}$$

is not the second-order relational coefficient. It lacks both the second-order inverse-coordinate displacement and the quadratic Taylor term.

Likewise, replacing the inverse map by $y - \kappa w - \kappa^2 v$ is incorrect. The term $\kappa^2 w^\nu \partial_\nu w^\mu$ in is forced by formal inversion.

Initial data are part of the observable

If ξ does not vanish near Σ_0, then

$$\mathrm{E}_0 \mathrm{Box}_\eta \xi = \xi - H_0[\xi]$$

rather than ξ. Similarly,

$$\mathrm{E}_m P_m(X_\xi \varphi_0) = X_\xi \varphi_0 - H_m[X_\xi \varphi_0].$$

Consequently, the boundary-data terms must be included in the gauge transformation. Their absence here is a proved consequence of the admitted support condition, not a universal identity.

The boundary of the quantum claim

No quantum products have been used

All products in the proofs above are products of smooth classical functions. A Green operator is applied to a smooth source; nested applications are compositions of well-defined Cauchy solution maps. This does not establish products of the associated distributional quantum kernels.

In particular, no microcausal functional domain, time-ordered extension, or renormalized retarded product for the harmonic insertion has been constructed.

The distinction matters because the coordinate prescription is part of the observable. Fröb's one-loop calculation of a relational scalar two-point function includes cancellations between ordinary self-energy contributions and coordinate corrections [5]. That calculation specializes the coordinate Green function to a Feynman prescription [5, Sections 2–3]. Its cancellation cannot be imported as a result for the present zero-Cauchy-data prescription.

The four terms of a possible successor construction

If a quantum retarded-product construction is subsequently established, and if the interaction and composite insertion have expansions

$$V = \kappa V_1 + \kappa^2 V_2 + \cdots, \qquad \mathcal{O} = \mathcal{O}^{(0)} + \kappa \mathcal{O}^{(1)} + \kappa^2 \mathcal{O}^{(2)} + \cdots,$$

then the coefficient of order κ^2 has the formal bookkeeping pattern

$$\mathcal{O}^{(2)} + R_1(V_1; \mathcal{O}^{(1)}) + R_1(V_2; \mathcal{O}^{(0)}) + \frac{1}{2} R_2(V_1, V_1; \mathcal{O}^{(0)}).$$

Here records obligations; it is not a definition of products whose existence has been proved.

A connected two-point response would additionally contain the product of the two first-order observable coefficients. A renormalized construction would have to name the state, the gravitational gauge-fixing and ghost sector, and the convention relating powers of $\hbar$ to graph loops.

The Action Ward Identity framework treats dependence on the action rather than on a representative Lagrangian density [3]. Applying that framework here would still require an extension to the declared harmonic-coordinate kernels. Integration by parts in the classical formulas is not a proof of that extension.

Why a counterterm catalogue is not yet available

At order κ^2, the classical formulas use at most two factors of the switch, counting a differentiated switch as one factor. This is insertion bookkeeping, not a proof of renormalization completeness.

A future response-level catalogue must fix, at minimum:

1. the response, state, and loop convention;

2. the action and composite normalization conditions;

3. bounds on field degree, derivative order, and switch degree;

4. the permitted mass dependence and constant coefficients;

5. the admitted terms associated with the initial surface;

6. the propagation of every admitted term through the same Cauchy kernels used for the reference response. No finite-dimensional ambiguity theorem follows merely from fixing the derivative order while permitting arbitrary functions of the switch.

Local ambiguities can have nonlocal responses

A useful classical identity makes one potential shortcut unavailable. Let $D_s = D + s\,\delta D$ have the same boundary prescription and let G_s be its zero-data inverse. Differentiating $D_s G_s = \mathrm{id}$ gives

$$\frac{d}{ds} G_s \Big|_{s=0} = -G\,\delta D\,G.$$

The solution of the differentiated equation has zero data, which justifies the inverse on the left.

Thus a local change of an equation can produce a propagated, nonlocal response. Disjoint observation supports do not by themselves remove all action-counterterm responses.

The effective-field-theory distinction between analytic local terms and nonanalytic long-distance loop contributions is a valuable guide [4], but it does not establish that every nonlocal smeared number is normalization independent. A separator must annihilate the complete propagated action-and-composite ambiguity space of the chosen response.

Status ledger

toprule Question Status in this paper

midrule First-order plateau cancellation Proved

Second-order classical intertwiner Proved

Restriction to $\mathscr{J}_2^{\beta}$ Proved

Coefficientwise causal support bound Proved

Classical fixed-data comparison of switches Proved

Quantum products for the harmonic insertion Not constructed

Quantum Ward identity Not proved

Renormalized quantum response Not constructed

Complete response-level ambiguity catalogue Not classified

Finite or kernel-level separation certificate Not evaluated

bottomrule

If the programme's renormalization-status terminology is used, the quantum-response status is *RG-open*. It is neither RG-separated nor RG-finite. No failure of quantum products is claimed either: lack of a construction is not an obstruction theorem.

Conclusion

For the action-level switching prescription , the first-order plateau repair extends through second order. The extension requires the genuine metric two-jet, the density-valued transformation law, the complete inverse harmonic-coordinate expansion, and fixed Cauchy data.

The mechanism is precise. On the support of an admitted gauge transformation the switched scalar operator is the

natural curved operator. Outside that support the transformation is the identity. The scalar solution and the harmonic-coordinate map therefore obey the same pullback law, and their relational composition is invariant.

The larger-support first-order obstruction remains intact. Equation identifies its defect, and Corollary explains why adding a second-order coefficient to the unchanged large-group construction cannot remove it.

What has been completed is the classical gate:

$$\text{boxed specified switching} + \mathscr{G}_U + \text{fixed Cauchy dressing} \quad \Rightarrow \quad \text{second-order classical intertwiner.}$$

A renormalized quantum response and its complete ambiguity space require further work on the repaired object. They are not consequences of the classical cancellation alone.

appendix

Sign, density, and switch audit

Euler sign

With signature $(-,+,+,+)$ and the scalar action , variation gives

$$\frac{\delta S_g^{\mathrm{sc}}}{\delta \varphi} = \mathrm{D}_g[\mathbf{g}]\varphi.$$

Since

$$\mathrm{D}_g[\mathbf{g}] = P_m + \kappa \mathrm{V}_g + \kappa^2 \mathrm{W}_g + \cdots,$$

the first-order equation is

$$P_m \varphi_{1,g} = -\mathrm{V}_g \varphi_0,$$

hence

$$\varphi_{1,g} = -\mathrm{E}_m \mathrm{V}_g \varphi_0.$$

If an alternative convention defines the perturbation operator with the opposite sign, the response formula must

change sign as well. The two conventions cannot be mixed.

The divergence term in a density transformation

For an orientation-preserving diffeomorphism,

$$F^{*}_{\mathrm{dens}} r = (\det DF)(r \circ F).$$

For the flow of ξ,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} F^{*}_{s,\,\mathrm{dens}} r = \xi^{\mu}\partial_{\mu} r + (\partial_{\mu}\xi^{\mu}) r.$$

This proves . Omitting the divergence term would replace the off-shell identity by an incorrect operator commutator. Its contribution disappears on a free scalar only after imposing $P_m \varphi_0 = 0$.

Switching before variation

For any smooth symmetric coefficient $A^{\mu\nu}$,

$$\partial_{\mu}(g A^{\mu\nu}\partial_{\nu}\psi) = g\partial_{\mu}(A^{\mu\nu}\partial_{\nu}\psi) + (\partial_{\mu} g) A^{\mu\nu}\partial_{\nu}\psi.$$

Therefore multiplying an unswitched equation by g is not the same as multiplying the interaction Lagrangian density by g before variation.

The main construction uses the latter prescription. This fixes both the switch derivatives and the second-order scalar operator.

Zero-data identities

For $\xi \in \mathfrak{g}_U$, both ξ and $X_\xi \varphi_0$ vanish on a neighbourhood of Σ_0. Accordingly,

$$\mathrm{E}_0 \mathrm{Box}_\eta\, \xi = \xi, \qquad \mathrm{E}_m P_m(X_\xi \varphi_0) = X_\xi \varphi_0.$$

The relevant fact is the vanishing Cauchy data. Compact support is a sufficient condition here because the admitted support is strictly to the future of Σ_0.

Direct cancellation of the second-order coefficient

For a vector field w, define the differential operator

$$J(w) = (w^\nu \partial_\nu w^\mu)\partial_\mu + \frac{1}{2} w^\mu w^\nu \partial_\mu \partial_\nu .$$

Then

$$A_2^{(g)} = \varphi_{2,g} - X_w \varphi_{1,g} - X_\nu \varphi_0 + J(w)\varphi_0 .$$

A direct expansion gives

$$J(w+\xi) - J(w) = X_w X_\xi + X_{X_\xi w} + \frac{1}{2} X_\xi^2 + \frac{1}{2} X_{X_\xi \xi} .$$

Here $X_\xi w$ denotes the vector with components $\xi^\nu \partial_\nu w^\mu$.

For completeness, the first-derivative part of the left-hand side is

$$(w^\nu \partial_\nu \xi^\mu + \xi^\nu \partial_\nu w^\mu + \xi^\nu \partial_\nu \xi^\mu)\partial_\mu ,$$

and its second-derivative part is

$$w^\mu \xi^\nu \partial_\mu \partial_\nu + \frac{1}{2} \xi^\mu \xi^\nu \partial_\mu \partial_\nu .$$

These are exactly the corresponding parts of .

Using –, the difference $A_2^{'} - A_2$ is

$$\begin{aligned} & X_\xi \varphi_{1,g} + \left(X_\zeta + \frac{1}{2} X_\xi^2 \right) \varphi_0 \\ & \quad - X_{w+\xi}(\varphi_{1,g} + X_\xi \varphi_0) + X_w \varphi_{1,g} \\ & \quad - X_{X_\xi w + \zeta + \frac{1}{2} X_\xi \xi} \varphi_0 + (J(w+\xi) - J(w))\varphi_0 . \end{aligned}$$

Terms containing $\varphi_{1,g}$ and ζ cancel immediately. The remaining operator on φ_0 is

$$-\frac{1}{2} X_\xi^2 - X_w X_\xi - X_{X_\xi w} - \frac{1}{2} X_{X_\xi \xi} + J(w+\xi) - J(w),$$

which vanishes by .

This calculation verifies the second-order relational cancellation without suppressing the quadratic terms in the finite gauge parameter.

Explicit Einstein-tensor coefficients

This appendix fixes the curvature convention used in Section . Write

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho}.$$

Define

$$\Gamma^{(1)\rho}_{\mu\nu}[h] = \frac{1}{2}\eta^{\rho\sigma}\big(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\big),$$

$$\Gamma^{(2)\rho}_{\mu\nu}[h,h] = -\frac{1}{2}h^{\rho\sigma}\big(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\big).$$

Then

$$\Gamma[\eta + \kappa h + \kappa^2 k] = \kappa\Gamma^{(1)}[h] + \kappa^2(\Gamma^{(1)}[k] + \Gamma^{(2)}[h,h]) + \mathcal{O}(\kappa^3).$$

The linear Ricci coefficient is

$$R^{(1)}_{\mu\nu}[h] = \partial_\rho \Gamma^{(1)\rho}_{\mu\nu}[h] - \partial_\nu \Gamma^{(1)\rho}_{\mu\rho}[h],$$

and the quadratic coefficient is

$$R^{(2)}_{\mu\nu}[h,h] = \partial_\rho \Gamma^{(2)\rho}_{\mu\nu}[h,h] - \partial_\nu \Gamma^{(2)\rho}_{\mu\rho}[h,h] + \Gamma^{(1)\rho}_{\rho\lambda}[h]\Gamma^{(1)\lambda}_{\mu\nu}[h] - \Gamma^{(1)\rho}_{\nu\lambda}[h]\Gamma^{(1)\lambda}_{\mu\rho}[h].$$

Set

$$r^{(1)}[h] = \eta^{\mu\nu}R^{(1)}_{\mu\nu}[h],$$

$$r^{(2)}[h,h] = \eta^{\mu\nu}R^{(2)}_{\mu\nu}[h,h] - h^{\mu\nu}R^{(1)}_{\mu\nu}[h].$$

Then

$$G^{(1)}_{\mu\nu}[h] = R^{(1)}_{\mu\nu}[h] - \frac{1}{2}\eta_{\mu\nu} r^{(1)}[h],$$

$$G^{(2)}_{\mu\nu}[h,h] = R^{(2)}_{\mu\nu}[h,h] - \frac{1}{2}\eta_{\mu\nu} r^{(2)}[h,h] - \frac{1}{2} h_{\mu\nu} r^{(1)}[h].$$

These formulas make Definition independent of any external convention for quadratic Einstein coefficients.

As a check, substituting $h_{\mu\nu} = 2\sigma\eta_{\mu\nu}$ into yields

$$G^{(1)}_{\mu\nu}[2\sigma\eta] = -2\partial_\mu\partial_\nu\sigma + 2\eta_{\mu\nu}\mathrm{Box}_\eta\,\sigma,$$

the identity used in Example .

Finally,

$$\partial^\mu T^{(0)}_{\mu\nu}[\varphi_0] = (P_m\varphi_0)\partial_\nu\varphi_0 = 0.$$

Thus the leading scalar source obeys the necessary flat conservation identity. This observation does not authorize zero metric Cauchy data for an arbitrary nonzero source: the gravitational constraint equations remain part of the choice of h and k.

A compact list of proved dependencies

The main result uses only the following ingredients:

1. the explicit action ;

2. coefficientwise Cauchy existence and uniqueness;

3. the support condition $\overline{U} \subset P_g$;

4. the naturality of the unswitched scalar density equation;

5. formal inversion of a map with leading term id;

6. the full second-order metric pullback –. No quantum product, counterterm classification, earlier-series

uniqueness theorem, or finite response matrix is used.

The Einstein–scalar restriction additionally uses the explicit coefficient definition in Appendix and the naturality of the Einstein tensor. It is not needed for the off-shell classical intertwiner.

Book expansion notes (t433u)

This chapter belongs to the book *Observation Boundaries and Finite Instruments: Series VII*. The first-order defect for larger gauge supports and the second-order classical intertwiner are retained. No higher coupling and no positive $\hbar$-order removes the larger-support first-order defect (Corollary); quantum products remain unconstructed. The defect is not repaired. Observation-boundary comparison is documented in [11]; the shredder interface is a source record in [12]; the Enter calculus journal record is [13].

Standing ledger. Not spacetime emergence; not Einstein dynamics; not a Page curve, island, bounce, vacuum-energy selection, or de Sitter state as an Enter theorem; not a derived metric. O.1 stays NOT_CLOSED. XVI-N is not discharged. XV-B is not opened. No quantum coefficient is admitted. Host C is not created. Dom (T) remains undetermined; J2 remains unresolved. Problems 6, 7, and 9 are named, not solved. Gate 0 and ST4b stay NOT_SUPPLIED. Host-A stays NAMED_COMPARISON_ONLY $\{\mathrm{GI}, \mathrm{CB}\}$. The shredder is not Gibbons–Hawking. Cited sheets are source records, not premises. Conjectures stay conjectures. wired=false.

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