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Landmark VII-B Cauchy harmonic insertions and second-order retarded products: a sharp-boundary obstruction in an off-shell polynomial domain

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Abstract

We investigate the first geometric coefficient of a relational scalar defined using harmonic coordinates with fixed zero Cauchy data. The quantum fields are a scalar and a gauge-fixed spin-two field, together with the ghost and auxiliary sectors of perturbative gravity. The proposed insertion space is the algebra of compactly supported microcausal polynomial kernels, with a fixed graded Hadamard contraction.

For the coordinate-response operator $L = \mathrm{E}_0 C$, we compute its distributional adjoint exactly. It contains a hypersurface delta distribution whose nonzero conormal covectors are timelike. We prove an exact criterion for a linearly smeared coordinate response to belong to the declared domain. More importantly, for every nonzero real test function supported strictly to the future of the initial surface, the first geometric insertion fails the microcausal condition. The proof exhibits a smooth scalar configuration at which the metric derivative has a nonvanishing hypersurface delta term. This excludes a coefficientwise, classically normalized quantization of that insertion in the declared off-shell algebra, and hence excludes the proposed termwise second-order construction in that algebra.

The obstruction is not a claim that the relevant distributions do not exist. Smooth-edge regularization is constructed and its sharp limit is analyzed. A smooth spacetime representative is also constructed for each linear

coordinate observable on the free mass shell. Neither result supplies the missing nonlinear interacting products. The paper therefore closes the boundary-domain alternative, not the second-order existence alternative.

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Question, result, and scope

A classical relational construction does not, by itself, specify a quantum product domain. This distinction is especially important when the reference coordinates are obtained by solving a hyperbolic equation with data prescribed on a sharp initial hypersurface.

We consider the geometric functional

$$\mathscr{B}_f[\mathbf{g},\varphi] = \int_M f(y)\,\varphi(\mathrm{X}_{\mathbf{g}}^{-1}(y))\,\mathrm{d}^4y,$$

where the four functions $X^{\mu}_{\mathbf{g}}$ are harmonic for **g**, with their values and first time derivatives fixed on an initial surface. All classical ingredients used below are specified explicitly in Section ; no theorem about their diffeomorphism intertwining is assumed or reproved.

The intended second-order interacting coefficient is formally

$$\mathscr{B}_f^{(2)} + R_1(V_1; \mathscr{B}_f^{(1)}) + R_1(V_2; \mathscr{B}_f^{(0)}) + \frac{1}{2} R_2(V_1, V_1; \mathscr{B}_f^{(0)}).$$

Before this expression can be constructed term by term, its geometric insertions must be admitted by the chosen domain. That prerequisite fails for the domain studied here.

The main result is Theorem . For every

$$0 \neq f \in C_c^{\infty}(\{t > 0\}; \mathbb{R}),$$

the first geometric coefficient

$$\mathscr{B}_f^{(1)}[h, \varphi] = -\int f(x)\, (\mathrm{E}_0 C h)^{\mu}(x)\, \partial_{\mu} \varphi(x)\ \mathrm{d}^4 x$$

is not a microcausal functional on the full smooth configuration space. The obstruction is a *proved wavefront component*, rather than a failure to verify a sufficient criterion for multiplying distributions.

We use a finite polynomial-kernel domain. This avoids relying on closure assertions for arbitrary smooth microcausal functionals: Hawkins, Rejzner, and Visser identify a closure problem for that larger class, introduce equicausal functionals, and establish equicausality for Wick polynomials with admissible fixed kernels [2]. The present obstruction is more elementary: it violates the microcausal cone condition itself, and therefore also excludes an equicausal representative that is equal to the original functional off shell.

The gravitational interpretation is deliberately limited. The gauge-fixed field content belongs to perturbative gravity, rather than to a scalar theory with a prescribed external metric. However, admission to a gauge-fixed algebra is not the same as physicality. The latter requires a separate gauge-cohomological analysis, as in the perturbative framework of Brunetti, Fredenhagen, and Rejzner [1].

What is and is not proved. The following statements are established:

1. an exact sharp-Cauchy adjoint formula;

2. an exact domain criterion for linearly smeared coordinate responses;

3. exclusion of every nontrivially smeared first geometric insertion from the declared off-shell domain;

4. impossibility of an off-shell, classically normalized quantization in that domain;

5. a smooth-edge approximation and a linear on-shell repair. No Ward identity, radiative response, classification of interaction counterterms, or second-order product-existence theorem is asserted. In particular, the result does not exclude a construction in an appropriately enlarged boundary algebra or an interacting on-shell formulation.

Classical input and geometric bookkeeping

Spacetime, fields, and zero-Cauchy response

Let

$$M = \mathbb{R}^{1,3}, \qquad \eta = \operatorname{diag}(1,-1,-1,-1), \qquad \Sigma_0 = \{t = 0\}, \qquad M_+ = \{t \geq 0\}.$$

Indices are raised and lowered with η, and

$$\operatorname{square} = \partial_t^2 - \Delta_{\boldsymbol{x}}.$$

The metric perturbation $h_{\mu\nu}$ is a smooth symmetric covariant tensor. Write

$$h = \eta^{\mu\nu} h_{\mu\nu}, \qquad C^\mu[h] = \partial_\nu h^{\nu\mu} - \frac{1}{2}\partial^\mu h.$$

The symbol h without indices denotes the trace only where no confusion can arise.

Let E_{R} and E_{A} denote the retarded and advanced Green operators for square, respectively:

$$\operatorname{square} \mathrm{E}_{\mathrm{R}}\, u = u, \qquad \operatorname{square} \mathrm{E}_{\mathrm{A}}\, u = u, \qquad E_{\mathrm{R}}^{*} = E_{\mathrm{A}}.$$

The zero-Cauchy solution operator E_0 is defined by

$$\operatorname{square} \mathrm{E}_0 s = s \quad (t > 0), \qquad (\mathrm{E}_0 s)|_{\Sigma_0} = 0, \qquad \partial_t (\mathrm{E}_0 s)|_{\Sigma_0} = 0.$$

For smooth s, it can be realized for $t > 0$ as

$$\mathrm{E}_0 s = E_{\mathrm{R}}(\theta s), \qquad \theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

The right-hand side is understood as the retarded solution of the past-compact distributional source θs. Its restriction to M_+ has the prescribed zero Cauchy data.

The first coordinate response is

$$w^{\mu}[h] = L^{\mu} h, \qquad L = \mathrm{E}_0 C.$$

This operator is not switched.

For a compact set $K \mathrm{Subset} \{t > 0\}$, set

$$\mathscr{D}_K = J^{-}(K) \cap M_+.$$

In Minkowski spacetime this is compact. Finite propagation speed implies that $w[h]|_K$ depends only on h in $\mathscr{D}_K$, including its trace on the initial surface.

Harmonic expansion

Introduce a single metric field by

$$\mathbf{g} = \eta + \kappa h.$$

Define

$$d(\kappa) = \sqrt{-\det(\eta + \kappa h)}, \qquad A^{\mu\nu}(\kappa) = d(\kappa)\, \mathbf{g}^{\mu\nu}.$$

Their first coefficients are

$$d_1 = \frac{1}{2} h, \qquad d_2 = \frac{1}{8} h^2 - \frac{1}{4} h_{\rho\sigma} h^{\rho\sigma},$$

$$A_1^{\mu\nu} = \frac{1}{2} h\, \eta^{\mu\nu} - h^{\mu\nu},$$

$$A_2^{\mu\nu} = h^{\mu}{}_{\rho} h^{\rho\nu} - \frac{1}{2} h\, h^{\mu\nu} + d_2 \eta^{\mu\nu}.$$

The harmonic-coordinate equations are written in density form:

$$\partial_\nu \left(A^{\nu\rho}(\kappa) \partial_\rho \mathrm{X}_{\mathbf{g}}^{\mu} \right) = 0.$$

Fix

$$\mathrm{X}_{\mathbf{g}}^{\mu}|_{\Sigma_0} = x^{\mu}|_{\Sigma_0}, \qquad \partial_t \mathrm{X}_{\mathbf{g}}^{\mu}|_{\Sigma_0} = \partial_t x^{\mu}|_{\Sigma_0}.$$

Writing

$$\mathrm{X}_{\mathbf{g}}^{\mu} = x^{\mu} + \kappa w^{\mu} + \kappa^2 v^{\mu} + O(\kappa^3)$$

gives and

$$v^{\mu}[h, h] = -\mathrm{E}_0 \left\{ \partial_\nu A_2^{\nu\mu} + \partial_\nu \left(A_1^{\nu\rho} \partial_\rho w^{\mu} \right) \right\}.$$

Here v denotes the coefficient of κ^2, without a factorial.

Formal inversion yields

$$(\mathrm{X}_{\mathbf{g}}^{-1})^{\mu} = x^{\mu} - \kappa w^{\mu} + \kappa^2 \left(w^{\nu} \partial_\nu w^{\mu} - v^{\mu} \right) + O(\kappa^3).$$

Consequently,

$$\mathscr{B}_f^{(0)}[h, \varphi] = \int f \varphi \, \mathrm{d}^4 x,$$

$$\mathscr{B}_f^{(1)}[h, \varphi] = -\int f w^{\mu} \partial_\mu \varphi \, \mathrm{d}^4 x,$$

$$\mathscr{B}_f^{(2)}[h, \varphi] = \int f \left[\left(w^{\nu} \partial_\nu w^{\mu} - v^{\mu} \right) \partial_\mu \varphi + \frac{1}{2} w^{\mu} w^{\nu} \partial_\mu \partial_\nu \varphi \right] \mathrm{d}^4 x.$$

These are geometric coefficients, expanded before any interacting field equations are solved.

No double counting

For clarity, consider arbitrary smooth formal curves

$$H_{\mathrm{int}} = h + \kappa k + O(\kappa^2), \qquad \varphi_{\mathrm{int}} = \varphi_0 + \kappa\varphi_1 + \kappa^2\varphi_2 + O(\kappa^3).$$

Then direct substitution gives

$$\begin{aligned}\mathscr{B}_f[\eta + \kappa H_{\mathrm{int}}, \varphi_{\mathrm{int}}] = {}& \int f\varphi_0 \, \mathrm{d}^4x + \kappa \int f\big(\varphi_1 - w[h]^\mu \partial_\mu \varphi_0\big)\, \mathrm{d}^4x \\ & + \kappa^2 \int f[\varphi_2 - w[h]^\mu \partial_\mu \varphi_1 - w[k]^\mu \partial_\mu \varphi_0 \\ & + \big(w[h]^\nu \partial_\nu w[h]^\mu - v[h,h]^\mu\big)\partial_\mu \varphi_0 \\ & + \frac{1}{2} w[h]^\mu w[h]^\nu \partial_\mu \partial_\nu \varphi_0]\, \mathrm{d}^4x + O(\kappa^3).\end{aligned}$$

Thus the second metric coefficient k enters through the interacting expansion of one graviton field. It is not a second independent quantum field.

Equation is an algebraic identity on smooth configurations. It is not a theorem asserting the existence of quantum retarded products. In particular, φ_1 and φ_2 are not included in the bare coefficients –.

Quantum model and declared insertion domain

Gauge-fixed free multiplet

The quantum field multiplet is

$$\Psi = (h_{\mu\nu}, \varphi, c^\mu, \bar{c}_\mu, \beta_\mu).$$

The fields $c, \bar{c}$ are odd; the remaining fields are even. The scalar has mass $m > 0$.

Let the trace-reversal operator be

$$(\mathrm{T}h)_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \qquad \mathrm{T}^2 = \mathrm{id}.$$

The free action, modulo an unswitched total divergence, is

$$S_0 = \int \Big[\frac{1}{2}\partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\lambda h\, \partial^\lambda h + \frac{1}{2}\partial_\lambda \varphi\, \partial^\lambda \varphi - \frac{1}{2} m^2 \varphi^2 - \bar{c}_\mu \mathrm{square}\, c^\mu - \frac{1}{4}\beta_\mu \beta^\mu \Big]\, \mathrm{d}^4x.$$

This is the linear de Donder gauge with a decoupled, shifted Nakanishi–Lautrup field. An explicit nonlinear action representative and its switched vertices are specified in Appendix .

With the metric used to identify dual tensor bundles, the free Hessian has diagonal even blocks

$$P_{hh} = -\mathrm{T}\mathrm{square}, \qquad P_{\varphi\varphi} = -(\mathrm{square} + m^{2}), \qquad P_{\beta\beta} = -\frac{1}{2}\eta,$$

and odd block, in the order $(c, \bar{c})$,

$$P_{\mathrm{odd}} = \begin{pmatrix} 0 & \mathrm{square} \\ -\mathrm{square} & 0 \end{pmatrix}.$$

The displayed odd signs correspond to writing the quadratic action as $-\bar{c}_{\mu}\,\mathrm{square}\,c^{\mu}$.

For $m \geq 0$, write

$$\Delta_{m}^{+}(x-y) = \int_{\mathbb{R}^{3}} \frac{\mathrm{d}^{3}\boldsymbol{k}}{(2\pi)^{3}\, 2\omega_{m}(\boldsymbol{k})} e^{-i\omega_{m}(\boldsymbol{k})(x^{0}-y^{0}) + i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}, \qquad \omega_{m}(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}|^{2} + m^{2}}.$$

Fix the graded two-point matrix by

$$\mathscr{H}_{hh\,\mu\nu\rho\sigma} = \mathrm{T}_{\mu\nu\rho\sigma}\Delta_{0}^{+}, \quad \mathscr{H}_{\varphi\varphi} = \Delta_{m}^{+},$$

$$\mathscr{H}^{c^{\mu}\bar{c}_{\nu}} = \delta^{\mu}_{\nu}\Delta_{0}^{+}, \qquad \mathscr{H}^{\bar{c}_{\nu}c^{\mu}} = -\delta^{\mu}_{\nu}\Delta_{0}^{+}.$$

All remaining blocks vanish. In particular, the algebraic auxiliary field has zero two-point function.

No positivity assertion is made for this gauge-fixed matrix. Moreover, gauge fixing does not impose $C[h] = 0$ as an identity on the off-shell configuration space.

The polynomial kernel domain

Let $\overline{V}_{\pm} \subset T^{*}M$ be the future/past nonspacelike covector cones, including the zero covector in each fibre. For $n \geq 1$, define

$$\Xi_{n} = T^{*}M^{n} \setminus \left(\overline{V}_{+}^{\,n} \cup \overline{V}_{-}^{\,n}\right).$$

Wavefront sets never contain the total zero covector.

Definition (Admissible polynomial kernels). The space $\mathcal{P}_{\mathrm{mc}}$ consists of finite polynomials in the field multiplet whose coefficient kernels are compactly supported distributions u_n satisfying

$$\mathrm{WF}(u_n) \subset \Xi_n.$$

The kernels have the graded symmetry appropriate to their field slots. In a purely even sector,

$$F(\Psi) = \sum_{n=0}^{N} \frac{1}{n!} \langle u_n, \Psi^{\otimes n} \rangle.$$

The quantum polynomial algebra is

$$\mathfrak{A}_{\mathrm{poly}} = \mathcal{P}_{\mathrm{mc}}[[\hbar]]$$

with the graded Wick product determined by $\mathscr{H}$.

For fixed polynomial degree the Wick expansion has only finitely many contractions. The usual wavefront calculus gives the Wick product on this domain; the polynomial setting also falls within the fixed-kernel results of [2]. Background on these contractions and their microlocal formulation can be found in [3].

We shall use only a necessary property of this domain.

Lemma (Necessary derivative condition). *If $F \in \mathcal{P}_{\mathrm{mc}}$, then every functional derivative satisfies*

$$\mathrm{WF}(F^{(r)}(\Psi)) \subset \Xi_r$$

at every smooth configuration Ψ. In particular, $F^{(1)}(\Psi)$ has no nonzero timelike covector in its wavefront set.

Proof. Differentiation produces finite sums of partial contractions of the compact kernels u_n with smooth configurations. In the wavefront bound for a partial contraction, the covectors in the contracted slots must be zero. If the remaining covectors were all future nonspacelike, or all past nonspacelike, adjoining the zero covectors would give a forbidden element of $\mathrm{WF}(u_n)$. This proves the claim. The same argument applies componentwise in the graded theory.

The geometric coefficients are initially considered as functionals on the full space of smooth configurations.

Quotienting by the free field equations is not part of Definition .

Definition (Off-shell classical normalization). An admissible quantization of a classical insertion F is an element

$$Q(F) = \sum_{r \geq 0} \hbar^{r} Q_{r} \in \mathfrak{A}_{\mathrm{poly}}$$

whose coefficient Q_{0} equals F on every smooth off-shell configuration.

This requirement distinguishes equality of the original functional from equality only after imposing field equations.

Exact sharp-Cauchy adjoint

Let

$$a \in C_{c}^{\infty}(\{t > 0\}; T^{*}M)$$

be a test covector, and define

$$\ell_{a}[h] = \int a_{\mu}(x)(Lh)^{\mu}(x)\ \mathrm{d}^{4}x, \qquad b_{\mu} = E_{\mathrm{A}} a_{\mu}.$$

The field b is smooth. It is supported in $J^{-}(\operatorname{supp} a)$, and it solves

$$\operatorname{square} b_{\mu} = a_{\mu}.$$

In a full neighbourhood of Σ_{0}, it solves the homogeneous wave equation.

The formal adjoint of C, using the pairing $\int b_{\mu} C^{\mu}[h]$, is

$$(C^{*}b)^{\alpha\beta} = -\partial^{(\alpha} b^{\beta)} + \frac{1}{2}\eta^{\alpha\beta}\partial^{\mu} b_{\mu}.$$

Proposition (Exact adjoint and boundary coefficient). *Let $n_{\mu} = \partial_{\mu} t$. The compactly supported distributional adjoint $q_{a} = L^{*} a$ is*

$$q_a^{\alpha\beta} = C^*(\theta b)^{\alpha\beta}$$

$$= \theta(C^* b)^{\alpha\beta} + \delta(t)\,\mathrm{B}^{\alpha\beta}(b|_{\Sigma_0}),$$

$$\mathrm{B}^{\alpha\beta}(v) = -n^{(\alpha}v^{\beta)} + \frac{1}{2}\eta^{\alpha\beta}n^{\mu}v_{\mu}.$$

Furthermore,

$$\operatorname{supp} q_a \subset J^-(\operatorname{supp} a) \cap M_+.$$

There is no $\delta'(t)$ term in this adjoint.

Proof. For smooth h, the retarded representation and formal adjointness give

$$\ell_a[h] = \langle a, E_{\mathrm{R}}(\theta Ch)\rangle = \langle E_{\mathrm{A}}a, \theta Ch\rangle = \langle C^*(\theta b), h\rangle.$$

The pairings involve only the compact causal set $J^-(\operatorname{supp} a) \cap M_+$. Hence the calculation is valid without a global compact-support assumption on h.

Applying the product rule in distributions and using

$$\partial_\mu \theta = n_\mu \delta(t)$$

gives . Since C^* has differential order one, it differentiates the step function at most once. This proves the absence of a delta derivative. Differentiation does not enlarge support, giving .

Lemma (Nondegeneracy of the boundary map). *The map $v \mapsto \mathrm{B}(v)$ is injective. In the chosen coordinates,*

$$\mathrm{B}^{00}(v) = -\frac{1}{2}v_0, \qquad \mathrm{B}^{0i}(v) = \frac{1}{2}v_i, \qquad \mathrm{B}^{ij}(v) = -\frac{1}{2}\delta^{ij}v_0.$$

Proof. The displayed components follow directly from . The 00 and $0i$ components recover all components of v.

Wavefront set

Let

$$N^*\Sigma_0 = \{(0, \boldsymbol{x}; \tau\ \mathrm{d}t) : \tau \neq 0\}.$$

Every element of this conormal bundle is timelike.

Lemma (Actual conormal singularities). *For every a as above,*

$$\mathrm{WF}\,(q_a) \subset N^{*}\Sigma_0.$$

If $b(0,\boldsymbol{x}_0) \neq 0$*, then*

$$(0,\boldsymbol{x}_0;\tau\ \mathrm{d}t) \in \mathrm{WF}\,(q_a) \qquad \text{for every} \tau \neq 0.$$

In particular, the hypersurface delta term cannot be cancelled by the step-function term in .

Proof. Both coefficients in are smooth. The standard wavefront bounds for a smooth multiple of a step or delta distribution therefore give ; see [4].

For the lower bound, choose a tensor component for which $\mathrm{B}(b(0,\boldsymbol{x}_0)) \neq 0$, possible by Lemma . Locally that component is

$$u(t,\boldsymbol{x}) = \delta(t)v(\boldsymbol{x}) + \theta(t)r(t,\boldsymbol{x}), \qquad v(\boldsymbol{x}_0) \neq 0,$$

with r, v smooth.

After compact localization, Fourier transformation in the normal variable has the form

$$\widehat{u}(\tau,\zeta) = \widehat{v}_{\mathrm{loc}}(\zeta) + O(|\tau|^{-1})$$

at fixed tangential frequency ζ, in either normal direction. For any localization nonzero at $(0,\boldsymbol{x}_0)$, the localized v is not identically zero, so its tangential Fourier transform is nonzero at some fixed ζ. Thus rapid decay fails in every conic neighbourhood of each nonzero normal covector. The step term has one lower normal Fourier order and cannot cancel the nonzero leading term.

An exact criterion for linear observables

Theorem (Linear domain criterion). *For* $a \in C_c^{\infty}(\{t>0\}; T^{*}M)$*, the following are equivalent:*

1. $\ell_a \in \mathcal{P}_{\mathrm{mc}}$*;*

2. *$q_a = L^* a$ is smooth;*

3. *the advanced solution $b = E_A a$ has vanishing full Cauchy data on Σ_0:*

$$b|_{\Sigma_0} = 0, \qquad \partial_t b|_{\Sigma_0} = 0.$$

When these conditions hold, q_a is a smooth compactly supported test tensor.

Proof. For a linear functional, domain membership is exactly the condition $\mathrm{WF}(q_a) \subset \Xi_1$. By , every possible singular covector of q_a is timelike. Hence domain membership is equivalent to smoothness.

Suppose q_a is smooth. Its delta coefficient must vanish: a distribution supported on the surface cannot cancel a locally bounded step term. Lemma gives $b|_{\Sigma_0} = 0$. Smoothness of the remaining distribution $\theta C^* b$ implies

$$(C^* b)|_{\Sigma_0} = 0.$$

Since the trace of b vanishes identically on the surface, all of its tangential derivatives vanish there. Formula then gives

$$(C^* b)|_{\Sigma_0} = \mathrm{B}(\partial_t b|_{\Sigma_0}).$$

Injectivity again yields $\partial_t b|_{\Sigma_0} = 0$.

Conversely, choose $\varepsilon > 0$ such that $a = 0$ for $t < \varepsilon$. On that region b solves the homogeneous wave equation. Vanishing Cauchy data imply, by uniqueness, that $b = 0$ for $t < \varepsilon$. Thus $\theta b = b$ is smooth, and $q_a = C^* b$ is smooth. Its compact support follows from .

Remark. The theorem does not say that a distribution with timelike wavefront set is intrinsically meaningless as a quantum smearing. It says that this smearing is excluded by the specific off-shell domain in Definition . Section constructs a linear on-shell representative.

Exclusion of the first geometric insertion

The obstruction for the bilinear insertion is stronger than the existence of exceptional bad test covectors. Every

nonzero real smearing f produces a failure somewhere in the smooth configuration space.

Lemma (A nonzero boundary trace for every nonzero smearing). *Let* $0 \neq f \in C_c^\infty(\{t > 0\}; \mathbb{R})$. *There exists* $\varphi_f \in C_c^\infty(M; \mathbb{R})$ *such that, for*

$$a_\mu = f\,\partial_\mu \varphi_f, \qquad b_\mu = E_{\mathrm{A}} a_\mu,$$

the trace $b_0|_{\Sigma_0}$ *is not identically zero. One explicit choice is*

$$\varphi_f = -\partial_t(tf).$$

Proof. Let

$$F(t) = \int_{\mathbb{R}^3} a_0(t, \boldsymbol{x})\ \mathrm{d}^3\boldsymbol{x}, \qquad B(t) = \int_{\mathbb{R}^3} b_0(t, \boldsymbol{x})\ \mathrm{d}^3\boldsymbol{x}.$$

On every bounded time interval, the spatial supports of a_0 and b_0 are contained in a common compact set.

Integrating square $b_0 = a_0$ over space therefore yields

$$B''(t) = F(t).$$

Since b_0 is advanced, $B = B' = 0$ for sufficiently large positive t. Consequently,

$$B(0) = \int_0^\infty tF(t)\ \mathrm{d}t = \int_M tf\,\partial_t \varphi_f\ \mathrm{d}^4x.$$

Set $u = \partial_t(tf)$, so that $\varphi_f = -u$. Integration by parts gives

$$B(0) = \int_M u^2\ \mathrm{d}^4x.$$

This number is strictly positive. Indeed, $u = 0$ would imply that tf is independent of t; compact time support would then force $f = 0$. Hence $b_0|_{\Sigma_0}$ cannot vanish identically.

Theorem (Sharp-boundary obstruction). *Fix the field multiplet and the off-shell domain of Section . For every* $0 \neq f \in C_c^\infty(\{t > 0\}; \mathbb{R})$*:*

1. $\mathscr{B}_f^{(1)}$ *is a continuous bilinear functional of smooth* h, φ*, with compact functional support contained in* $\mathscr{D}_{\operatorname{supp} f}$*,*

2. *It is not microcausal and, in particular,*

$$\mathscr{B}_f^{(1)} \notin \mathcal{P}_{\mathrm{mc}}.$$

3. *There is no admissible quantization of* $\mathscr{B}_f^{(1)}$ *satisfying Definition .*

4. *There is no termwise retarded-product construction of in* $\mathfrak{U}_{\mathrm{poly}}$ *that admits the original geometric insertion coefficientwise and preserves its off-shell classical normalization.*

Proof. Continuity of L on smooth fields, locally in the future half-space, follows from the continuous dependence of the inhomogeneous Cauchy problem on its source. Multiplication by f, differentiation of φ, and integration are continuous. This proves the first assertion, including polynomial smoothness. The support statement follows from finite propagation and .

For the second assertion, differentiate with respect to the metric field:

$$\frac{\delta \mathscr{B}_f^{(1)}}{\delta h}(h, \varphi) = -L^*(f\, \mathrm{d}\varphi).$$

This derivative is independent of h.

Choose $\varphi = \varphi_f$ from Lemma . Then $b = E_{\mathrm{A}}(f\, \mathrm{d}\varphi_f)$ has a nonzero trace at some point of Σ_0. Lemma shows that $-L^*(f\, \mathrm{d}\varphi_f)$ has nonzero timelike covectors in its wavefront set. This contradicts the necessary microcausal condition of Lemma .

If an admissible quantization existed, its coefficient of $\hbar^0$ would belong to $\mathcal{P}_{\mathrm{mc}}$ and would equal $\mathscr{B}_f^{(1)}$ on every smooth configuration. Differentiating this equality gives the same forbidden derivative, proving the third assertion.

Finally, a coefficientwise construction that admits the geometric insertion in $\mathfrak{U}_{\mathrm{poly}}[[\kappa]]$ must admit its coefficient $\mathscr{B}_f^{(1)}$, before applying $R_1(V_1; \cdot)$. The third assertion excludes this prerequisite. This proves the fourth assertion under its stated termwise and off-shell requirements.

Corollary (The problem is already in the quadratic kernel). *The compact bilinear coefficient kernel of* $\mathscr{B}_f^{(1)}$

does not satisfy the cone condition Ξ_2.

Proof. If that kernel satisfied Ξ_2, contraction of its scalar slot with the smooth field φ_f would produce a metric kernel satisfying Ξ_1, by the proof of Lemma . This contradicts and Lemma .

Thus the obstruction is not an artefact of evaluating a nonlinear functional far from the zero configuration. The fixed degree-two kernel itself fails the required wavefront condition. The witness configuration can also be multiplied by any nonzero real constant, including arbitrarily small ones.

What local changes cannot accomplish

Proposition (Changes supported away from the initial surface). *Let* Q_f *be a smooth functional such that, for every smooth configuration, its metric derivative is supported in a fixed closed set disjoint from a neighbourhood of* Σ_0. *Then*

$$\mathscr{B}_f^{(1)} + Q_f$$

is not microcausal.

In particular, adding a bilinear kernel supported on the ordinary diagonal, with its scalar projection contained in supp *f, cannot remove the obstruction.*

Proof. At the witness configuration, the metric derivative of Q_f vanishes in a neighbourhood of Σ_0. Therefore the derivative of the sum has exactly the same nonzero boundary wavefront component as .

For the final statement, a diagonal-supported kernel with scalar projection in supp *f* also has its metric projection in supp *f*, which is a positive distance in time from Σ_0.

Remark (Normal ordering and quantum corrections). Changing a prescription only in positive powers of $\hbar$ cannot change the contradiction in the classical coefficient. Allowing a new order-$\hbar^0$ boundary term changes the off-shell insertion unless that term vanishes as a functional. Theorem does not prohibit changing the observable; it prohibits calling such a change an off-shell quantization of the unchanged observable under Definition .

Two precise repairs and their limitations

Smooth-edge approximation

Choose a smooth function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(s) = 0 \quad (s \le 0), \qquad \chi(s) = 1 \quad (s \ge 1),$$

and set

$$\theta_\varepsilon(t) = \chi(t/\varepsilon), \qquad \varepsilon > 0.$$

Define

$$L_\varepsilon h = E_{\mathrm{R}}(\theta_\varepsilon C h), \qquad q_{a,\varepsilon} = C^{*}(\theta_{\mathrm{epsilonE}} \mathrm{A} a).$$

Proposition (Linear smoothing and the sharp limit). *For each fixed a:*

1. *$q_{a,\varepsilon}$ is a smooth compactly supported test tensor, and $\ell_{a,\varepsilon}[h] = \langle q_{a,\varepsilon}, h\rangle$ is a regular linear element of $\mathcal{P}_{\mathrm{mc}}$;*

2. *the supports of $q_{a,\varepsilon}$ are contained in the fixed compact set $J^{-}(\operatorname{supp} a) \cap M_{+}$;*

3. *$q_{a,\varepsilon} \to q_a$ in the strong topology of compactly supported distributions;*

4. *if $b|_{\Sigma_0} \neq 0$, this sequence cannot converge to an element of any fixed prescribed-wavefront space $\mathscr{E}'_\Gamma$ with $\Gamma \subset \Xi_1$.*

Proof. Smoothness and support follow directly from . For a smooth test tensor h,

$$\langle q_{a,\varepsilon} - q_a, h\rangle = \langle (\theta_\varepsilon - \theta) b, Ch\rangle.$$

The integrand is supported in a slab of time thickness ε, inside a common spatial compact set. It follows that

$$|\langle q_{a,\varepsilon} - q_a, h\rangle| \le C_a \varepsilon \sup_{\mathscr{D}_{\operatorname{supp} a}} |Ch|.$$

The estimate is uniform on bounded sets of smooth test tensors, which proves strong distributional convergence.

If convergence held in $\mathscr{E}'_\Gamma$ to an element of that space, continuity of its inclusion into distributions would identify the limit with q_a. But Lemma gives a timelike wavefront component of q_a, incompatible with $\Gamma \subset \Xi_1$.

The prescribed-wavefront topology in the last assertion can be taken to be the normal topology used for distributional operations in [5]. The assertion only requires that its inclusion into distributions be continuous.

At fixed ε, the source equation is

$$\square L_{\varepsilon} h = \theta_{\varepsilon} C h,$$

rather than the unswitched equation on the whole future half-space. Thus a fixed nonzero edge width changes the coordinate prescription. Proposition establishes a distributional sharp limit for linear smearings, not a sharp limit inside the declared insertion domain and not a theorem for the bilinear composite or its retarded products.

A linear on-shell spacetime representative

The failure of off-shell membership does not preclude an on-shell representation by smooth test tensors.

Let

$$E = E_{\mathrm{R}} - E_{\mathrm{A}}$$

be the causal propagator of square, acting componentwise.

Lemma (Smoothing by the causal propagator). *If q is a compactly supported tensor distribution with*

$$\mathrm{WF}(q) \subset N^{*}\Sigma_{0},$$

then $u = Eq$ is a smooth, spacelike-compact solution of $\square u = 0$.

Proof. The causal propagator extends to compactly supported distributions. Its distribution kernel has only paired characteristic covectors: both nonzero covectors in each wavefront pair are null. In the kernel-composition wavefront bound, a singularity in Eq would require a null input covector from $\mathrm{WF}(q)$, or a zero input covector in the wavefront set of the propagator kernel. Neither occurs: $\mathrm{WF}(q)$ is timelike, and the causal propagator has no such zero-input wavefront component. Hence Eq is smooth. This is an application of the standard kernel wavefront calculus reviewed in [4, 3].

The bisolution identity gives $\square Eq = 0$, and the support bound

$$\operatorname{supp} Eq \subset J(\operatorname{supp} q)$$

gives spacelike compactness.

Theorem (Linear on-shell repair). *Let $q = q_a = L^{*}a$. There exist a smooth compactly supported tensor q_{sm} and a compactly supported tensor distribution r such that*

$$q - q_{\mathrm{sm}} = \operatorname{square} r.$$

Consequently,

$$\langle q, h\rangle = \langle q_{\mathrm{sm}}, h\rangle$$

for every smooth solution of $\operatorname{square} h = 0$.

A construction is obtained by setting $u = Eq$, choosing a smooth time function $\rho(t)$ equal to zero sufficiently far in the past and one sufficiently far in the future, and defining

$$q_{\mathrm{sm}} = [\operatorname{square}, \rho]u = \rho'' u + 2\rho' \partial_t u.$$

Proof. Lemma gives a smooth, spacelike-compact homogeneous solution u. The derivatives of ρ have compact time support. Therefore q_{sm} is smooth and compactly supported.

The identities

$$E_{\mathrm{R}} q_{\mathrm{sm}} = \rho u, \qquad E_{\mathrm{A}} q_{\mathrm{sm}} = (\rho - 1)u$$

follow from the wave equation and uniqueness with zero data in the appropriate time direction. Thus

$$Eq_{\mathrm{sm}} = u = Eq.$$

Set $v = q - q_{\mathrm{sm}}$. Then v is compactly supported and $Ev = 0$, so

$$r := E_{\mathrm{R}} v = E_{\mathrm{A}} v.$$

Its support is contained in

$$J^{+}(\operatorname{supp} v) \cap J^{-}(\operatorname{supp} v),$$

a compact set. Moreover, square $r = v$, proving . Pairing that identity with a smooth solution h and using the formal self-adjointness of square proves .

Remark (Support and scope of the repair). The construction gives compact support, but it does not assert that q_{sm} has support in the original $\mathscr{D}_{\operatorname{supp} a}$. More importantly, is an equality on the free mass shell, not on the full configuration space.

For the gauge-fixed graviton in , the free equation is Tsquare $h = 0$, equivalent to square $h = 0$. Thus Theorem applies to the linear graviton observable. It does not construct an off-shell polynomial representative of the bilinear insertion, and it does not control contact terms in interacting retarded products.

Linear coordinate two-point pairings

For each fixed pair of coordinate tests a, a', the preceding construction also gives well-defined free on-shell pairings

$$\mathscr{H}_{wh}(a, t) := \mathscr{H}_{hh}(q_{\mathrm{sm},a}, t),$$

$$\mathscr{H}_{ww}(a, a') := \mathscr{H}_{hh}(q_{\mathrm{sm},a}, q_{\mathrm{sm},a'}),$$

where t is a smooth compactly supported graviton test tensor. These expressions are independent of the time cutoff used to construct the smooth representatives.

Indeed, two such representatives have a smooth compactly supported difference d with $Ed = 0$. Then

$$d = \operatorname{square}(E_{\mathrm{R}} d),$$

and $E_{\mathrm{R}} d = E_{\mathrm{A}} d$ is smooth and compactly supported. The bisolution property of $\mathscr{H}_{hh}$ annihilates this difference in either argument.

Equations – are statements about smooth-test pairings of free linear observables. They are not a licence to multiply the corresponding unsmeared kernels with scalar propagators, or to pull them back to the coincidences required by

a composite insertion.

Refusal of an exact smooth off-shell replacement

Proposition. *If q_a is singular, there is no smooth compactly supported tensor s such that*

$$\langle q_a, h\rangle = \langle s, h\rangle$$

for every smooth h.

Proof. The equality would in particular hold for all compactly supported smooth h, and hence would imply $q_a = s$ as distributions, contradicting singularity.

Thus the on-shell repair and the off-shell obstruction are compatible statements, with different equality requirements.

Consequences for second-order retarded products

Suppose

$$V = \kappa V_1 + \kappa^2 V_2 + O(\kappa^3)$$

and a retarded map is normalized formally by

$$R_V(F) = F + R_1(V;F) + \frac{1}{2}R_2(V,V;F) + \cdots.$$

Substituting the geometric expansion produces

$$R_V(\mathscr{B}_f) = \mathscr{B}_f^{(2)} + R_1(V_1;\mathscr{B}_f^{(1)}) + R_1(V_2;\mathscr{B}_f^{(0)}) + \frac{1}{2}R_2(V_1,V_1;\mathscr{B}_f^{(0)}).$$

This is a formal coefficient identity conditional on the existence of its terms.

The present result stops before the first mixed product:

$$R_1(V_1^{\mathrm{sc}};\mathscr{B}_f^{(1)}).$$

Its proposed geometric argument is not an element of the declared off-shell domain. No graph inventory or

extension prescription for this product is therefore claimed here.

toprule Item Established status

midrule Free multiplet and vertices Specified, including ghosts and auxiliaries; see Section and Appendix .

Geometric bookkeeping Equations – and ; no double counting.

Linear Cauchy adjoint Computed exactly; its domain criterion is Theorem .

First geometric composite Excluded from the declared off-shell domain by Theorem .

Mixed and two-vertex products Not constructed; the stated termwise off-shell route is blocked at its insertion-domain prerequisite.

Repairs Linear smoothing and a free on-shell representative are proved. No nonlinear interacting repair is asserted.

bottomrule

The obstruction is not a failure of a diagonal extension theorem. The distribution $L^{*}a$ already exists, with compact support and finite transverse singular order. It fails the selected wavefront condition. Distribution-extension methods such as those of Brunetti and Fredenhagen [6], and local covariant time-ordered products of local Wick polynomials as constructed by Hollands and Wald [7], do not by themselves place this nonlocal sharp-Cauchy insertion in the declared domain.

Nor is the result an off-diagonal multiplication obstruction: no claim is made that a characteristic product of a coordinate kernel and a quantum two-point function is intrinsically undefined. Such a claim would require an additional analysis of the actual tensor-valued graph distributions.

The precise conclusion is therefore

$$\text{boxed}\begin{matrix}\text{sharp zero-Cauchy dressing + exact off-shell classical normalization}\\ \text{+the polynomial microcausal domain}\end{matrix}\quad\Rightarrow\quad\text{failure at}\mathscr{B}_f^{(1)}\quad(f\neq 0).$$

A continuation must change at least one of these three requirements and prove that the new formulation preserves the intended observable in its stated sense.

No connected two-point response is declared in this paper. Accordingly, $\mathscr{B}_f^{(1)} \star \mathscr{B}_f^{(1)}$ is not constructed. The boundary result neither computes a quantum response nor settles a later Ward or renormalization problem.

appendix

An explicit action and vertex ledger

This appendix fixes a perturbative Einstein–scalar model with linear de Donder gauge fixing. Its purpose is to identify the quantum continuation unambiguously. None of the vertices is needed for the boundary obstruction, which is already present in the geometric insertion.

Fixed gravitational density representative

For $\mathbf{g} = \eta + \kappa h$, let

$$\Gamma^{\rho}_{\mu\nu}(\mathbf{g}) = \frac{1}{2}\mathbf{g}^{\rho\lambda}\big(\partial_\mu \mathbf{g}_{\lambda\nu} + \partial_\nu \mathbf{g}_{\lambda\mu} - \partial_\lambda \mathbf{g}_{\mu\nu}\big).$$

Choose the first-derivative Einstein density representative

$$\mathscr{L}_{\mathrm{grav}}(\kappa) = -\frac{2}{\kappa^2} A^{\mu\nu}(\kappa)\Big(\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\Big).$$

The curvature-sign convention is the one for which this is the Einstein density modulo a total divergence. The representative , rather than its equivalence class modulo divergences, is used before switching.

Because $\Gamma = O(\kappa)$, the density has a regular formal expansion

$$\mathscr{L}_{\mathrm{grav}}(\kappa) = \mathscr{L}_{\mathrm{grav},0} + \kappa \mathscr{L}_{\mathrm{grav},1} + \kappa^2 \mathscr{L}_{\mathrm{grav},2} + O(\kappa^3).$$

Its quadratic part, modulo an unswitched divergence, is

$$\mathscr{L}_{\mathrm{grav},0} = \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - (\partial_\mu h^{\mu\nu})(\partial^\rho h_{\rho\nu})$$

$$+(\partial_\mu h^{\mu\nu})\partial_\nu h - \frac{1}{2}\partial_\lambda h\,\partial^\lambda h.$$

Gauge fixing, ghosts, and auxiliary field

For an odd vector field c^{μ}, define

$$(\mathrm{D}_{\kappa}(h)c)_{\mu\nu} = \partial_{\mu} c_{\nu} + \partial_{\nu} c_{\mu}$$

$$+\kappa\left(c^{\rho}\partial_{\rho} h_{\mu\nu} + h_{\rho\nu}\partial_{\mu} c^{\rho} + h_{\mu\rho}\partial_{\nu} c^{\rho}\right).$$

The notation $\mathrm{D}_{\kappa}(h)$ here denotes the infinitesimal metric gauge generator, not an Euler operator.

With an unshifted auxiliary field B_{μ}, choose

$$\mathscr{L}_{\mathrm{gf}+\mathrm{gh}} = B_{\mu} C^{\mu}[h] - \frac{1}{4} B_{\mu} B^{\mu} - \bar{c}_{\mu} C^{\mu}[\mathrm{D}_{\kappa}(h)c].$$

The identity

$$C^{\mu}[\partial c + \partial c] = \mathrm{square}\, c^{\mu}$$

gives

$$\mathscr{L}_{\mathrm{gf}+\mathrm{gh}} = C_{\mu}[h] C^{\mu}[h] - \frac{1}{4}\beta_{\mu}\beta^{\mu} - \bar{c}_{\mu}\,\mathrm{square}\, c^{\mu} - \kappa \bar{c}_{\mu} C^{\mu}[\mathscr{L}_{c} h], \qquad \beta_{\mu} = B_{\mu} - 2C_{\mu}[h].$$

Adding $C_{\mu} C^{\mu}$ to gives the graviton quadratic density in , modulo an unswitched divergence. The shifted auxiliary field has no interaction vertices in this parametrization.

The chosen action is sufficient to define the gauge-fixed perturbative field content. No switched BRST or Ward identity is inferred from these formulas.

Scalar density and switching

Let

$$\mathscr{L}_{\mathrm{sc}}(\mathbf{g}, \varphi) = \frac{1}{2} A^{\mu\nu}(\kappa)\partial_{\mu}\varphi\, \partial_{\nu}\varphi - \frac{1}{2} m^{2} d(\kappa)\varphi^{2}.$$

Choose real compactly supported switches $g, g_{\mathrm{grav}}, g_{\mathrm{gh}}$. The scalar interaction is

$$\int g\left(\mathscr{L}_{\mathrm{sc}}(\eta + \kappa h, \varphi) - \mathscr{L}_{\mathrm{sc}}(\eta, \varphi)\right) \mathrm{d}^{4} x.$$

Thus one switch multiplies the complete curved-minus-flat density. The quadratic metric vertex is multiplied by g, not by g^2.

The full switched interaction is

$$V(\kappa) = \int g \left(\mathscr{L}_{\mathrm{sc}}(\eta + \kappa h, \varphi) - \mathscr{L}_{\mathrm{sc}}(\eta, \varphi) \right) \mathrm{d}^4 x$$

$$+ \int g_{\mathrm{grav}} \left(\mathscr{L}_{\mathrm{grav}}(\kappa) - \mathscr{L}_{\mathrm{grav},0} \right) \mathrm{d}^4 x$$

$$- \kappa \int g_{\mathrm{gh}} \, \bar{c}_{\mu} C^{\mu} [\mathscr{L}_{c} h] \, \mathrm{d}^4 x.$$

All representatives in this formula are fixed before multiplying by their switches.

In particular,

$$V_1^{\mathrm{sc}} = \frac{1}{2} \int g \left(A_1^{\mu\nu} \partial_{\mu} \varphi \, \partial_{\nu} \varphi - m^2 d_1 \varphi^2 \right) \mathrm{d}^4 x,$$

$$V_2^{\mathrm{sc}} = \frac{1}{2} \int g \left(A_2^{\mu\nu} \partial_{\mu} \varphi \, \partial_{\nu} \varphi - m^2 d_2 \varphi^2 \right) \mathrm{d}^4 x,$$

$$V_1^{\mathrm{gh}} = - \int g_{\mathrm{gh}} \, \bar{c}_{\mu} C^{\mu} [\mathscr{L}_{c} h] \, \mathrm{d}^4 x, \qquad V_2^{\mathrm{gh}} = 0,$$

$$V_1^{\mathrm{aux}} = 0, \qquad V_2^{\mathrm{aux}} = 0.$$

Finite formulas for the gravitational vertices

The following formulas specify the gravitational vertices without suppressing integrations by parts.

Define

$$T_{\lambda\mu\nu}(h) = \partial_{\mu} h_{\lambda\nu} + \partial_{\nu} h_{\lambda\mu} - \partial_{\lambda} h_{\mu\nu}.$$

Write

$$\Gamma^{\rho}_{\mu\nu} = \kappa \gamma^{\rho}_{1\mu\nu} + \kappa^2 \gamma^{\rho}_{2\mu\nu} + \kappa^3 \gamma^{\rho}_{3\mu\nu} + O(\kappa^4),$$

where

$$\gamma^{\rho}_{r\mu\nu} = \frac{1}{2} (-1)^{r-1} (h^{r-1})^{\rho}_{\sigma} \eta^{\sigma\lambda} T_{\lambda\mu\nu}(h), \qquad r = 1, 2, 3,$$

and $h^{0} = \mathrm{id}$.

For two Christoffel-type arrays P, Q, set

$$\mathscr{Q}_{\mu\nu}(P,Q) = P^{\rho}{}_{\mu\sigma}Q^{\sigma}{}_{\nu\rho} - P^{\rho}{}_{\mu\nu}Q^{\sigma}{}_{\rho\sigma}.$$

Then

$$\begin{aligned}\mathscr{L}_{\mathrm{grav},1} &= -2[A_1^{\mu\nu}\mathscr{Q}_{\mu\nu}(\gamma_1,\gamma_1)\\ &\qquad +\eta^{\mu\nu}(\mathscr{Q}_{\mu\nu}(\gamma_1,\gamma_2)+\mathscr{Q}_{\mu\nu}(\gamma_2,\gamma_1))],\\ \mathscr{L}_{\mathrm{grav},2} &= -2[A_2^{\mu\nu}\mathscr{Q}_{\mu\nu}(\gamma_1,\gamma_1)\\ &\qquad +A_1^{\mu\nu}(\mathscr{Q}_{\mu\nu}(\gamma_1,\gamma_2)+\mathscr{Q}_{\mu\nu}(\gamma_2,\gamma_1))\\ &\qquad +\eta^{\mu\nu}(\mathscr{Q}_{\mu\nu}(\gamma_1,\gamma_3)+\mathscr{Q}_{\mu\nu}(\gamma_2,\gamma_2)+\mathscr{Q}_{\mu\nu}(\gamma_3,\gamma_1))].\end{aligned}$$

Finally,

$$V_j^{\mathrm{grav}} = \int g_{\mathrm{grav}}\,\mathscr{L}_{\mathrm{grav},j}\,\mathrm{d}^4x, \qquad j = 1,2.$$

These expressions are finite local polynomials with smooth compact switches. Their local coefficient kernels satisfy the microcausal cone condition: the covectors in a diagonal conormal sum to zero, and hence cannot all lie in the same nonspacelike cone unless they all vanish. Derivatives do not enlarge that wavefront bound.

Kernel support and the location of the obstruction

For $x \in \{t > 0\}$, let $\Lambda^{\mu,\alpha\beta}(x,z)$ be the distribution kernel of L:

$$(Lh)^{\mu}(x) = \int \Lambda^{\mu,\alpha\beta}(x,z)h_{\alpha\beta}(z)\,\mathrm{d}^4z.$$

Equivalently, it is characterized by

$$\int a_{\mu}(x)\Lambda^{\mu,\alpha\beta}(x,z)\,\mathrm{d}^4x = [C^{*}(\mathrm{thetaE}_{\mathrm{A}}\,a)]^{\alpha\beta}(z).$$

The first geometric insertion has mixed kernel

$$K_f^{\alpha\beta}(z,x) = \partial_{x^\mu}\big(f(x)\Lambda^{\mu,\alpha\beta}(x,z)\big),$$

so that

$$\mathscr{B}_f^{(1)}[h,\varphi] = \langle K_f, h\otimes\varphi\rangle.$$

Its support satisfies

$$\operatorname{supp} K_f \subset \{(z,x) : x \in \operatorname{supp} f, \quad z \in J^-(x), \quad z^0 \geq 0\}.$$

This is a compact subset of $M \times M$.

The boundary part is located at $z^0 = 0$, whereas $x^0 > 0$ on $\operatorname{supp} f$. It is therefore away from the ordinary diagonal $z = x$. Corollary proves the failure of its required wavefront cone without needing to assert a complete wavefront equality for .

For the contracted adjoint kernel

$$q_a = \theta C^* b + \delta(t)\mathrm{B}(b|_{\Sigma_0}),$$

the transverse scaling degree of the delta term is one wherever its coefficient is nonzero. The step term has transverse scaling degree at most zero. These are finite-order distributions. Thus the obstruction is not excessive transverse scaling and is not the nonexistence of a distributional extension: it is incompatibility with the off-shell cone condition.

Logical boundary of the result

The proof separates four assertions that should not be identified:

1. The sharp-Cauchy insertion is a well-defined smooth polynomial functional on smooth classical fields.

2. Its fixed kernels are compactly supported distributions.

3. Those kernels belong to a prescribed quantum insertion domain.

4. Retarded products with those kernels exist in a common normalization prescription. The first two assertions hold for $\mathscr{B}_f^{(1)}$. The third fails in the domain of Definition . The fourth is therefore not reached by the stated termwise construction.

Theorem establishes that the analogous failure for a *linear* coordinate observable can be bypassed after restricting to the free equations of motion. It does not supply the nonlinear extension of that bypass. In particular, replacing a distributional metric smearing by an on-shell equivalent smooth smearing inside an interacting composite requires an additional proof controlling field dependence, support, and equation-of-motion contact terms.

The resulting manuscript is a closed boundary-domain result. It leaves the existence of a suitably repaired second-order relational insertion open.

Book expansion notes (t433u)

This sharp-boundary obstruction coexists with the VII-A plateau intertwiner [13]; coexistence is not cancellation. The obstruction is not removed by the finite part constructed in VII-D [15]. The mixed-Wick obstruction of VII-C [14] remains a separate domain question. Observation-boundary source record: [16].

Standing ledger. Not spacetime emergence; not Einstein dynamics; not a Page curve, island, bounce, vacuum-energy selection, or de Sitter state as an Enter theorem; not a derived metric. O.1 stays NOT_CLOSED. XVI-N is not discharged. XV-B is not opened. No quantum coefficient is admitted. Host C is not created. Dom (T) remains undetermined; J2 remains unresolved. Problems 6, 7, and 9 are named, not solved. Gate 0 and ST4b stay NOT_SUPPLIED. Host-A stays NAMED_COMPARISON_ONLY $\{\mathrm{GI}, \mathrm{CB}\}$. The shredder is not Gibbons–Hawking. Cited sheets are source records, not premises. Conjectures stay conjectures. wired=false.

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