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Finite part of the regulated Cauchy-reflection triangle

A strong numerical distributional limit for the VII-C family

Ngo Cheung

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Abstract

For the insertion-kernel regularization fixed in VII-C, we prove existence of the strong distributional finite part of its numerical reflection-triangle coefficient. The proof applies to arbitrary compact joint tests. An explicit normalization ledger identifies the leading reflected contribution with the prescribed light-ray residue, including its density and every Fourier factor. A joint spatial Fourier argument gives a regulator-independent ultraviolet decomposition: the radial coefficient differs from that residue by $O(r^{-1})$ in a fixed finite-order distribution seminorm, and the remaining regulator-placement error is $O(\varepsilon(1 + |\log \varepsilon|))$. The estimates retain the full phase and are uniform through the intersection of reflected collinearity with the first–third observation diagonal. For two admissible normalized radial mollifiers, the complete conversion is $c_{ab}r$, with an explicit real coefficient, as a corollary of the ultraviolet decomposition. The result is a numerical subtraction prescription, not an algebra construction.

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Scope and fixed data

Let $M = \mathbb{R}^{1+3}$, $M_{+} = \{t > 0\}$, and $X = M_{+}^{3}$. We ask whether

$$u_{\mu}^{\rho} = \lim_{\varepsilon \text{downarrow } 0} \left(T_{\varepsilon}^{\rho} - \log\left(\frac{1}{\mu\varepsilon}\right) r \right), \qquad \mu > 0,$$

exists strongly in $\mathscr{D}'(X)$, with the counterterm class fixed to

$$\mathfrak{C} = \mathbb{R} r.$$

The family T_{ε}^{ρ} is precisely the numerical VII-C family: both slots of each insertion are regularized with the same fixed mollifier ρ, while the single middle observable is unsmeared.

The formulas on the following convention sheet are from 085T_Landmark_VIIC_v1.tex [1]. These displayed formulas define the family studied in this paper. The proof requires no companion methods manuscript and no unspecified normalization.

We reserve μ for the subtraction scale and write θ for the mollifier profile denoted by μ in VII-C. The symbol g denotes the middle test; $s = y^{0}$ is a time coordinate.

Source convention sheet

begingroup 4pt 4pt 3pt 3pt

Insertion and placement (eq:B, eq:C, eq:Beps in VII-C).

$$\eta = \operatorname{diag}(-1,1,1,1), \qquad \operatorname{square} = -\partial_t^2 + \Delta, \qquad (Ch)^\mu = \partial_\nu h^{\mu\nu} - \tfrac{1}{2}\partial^\mu(\eta^{\alpha\beta}h_{\alpha\beta}), \qquad L = \mathrm{E}_0 C,$$
$$\operatorname{square}\mathrm{E}_0 j = j, \qquad (\mathrm{E}_0 j)|_{t=0} = 0, \qquad \partial_t(\mathrm{E}_0 j)|_{t=0} = 0,$$
$$B_f = -\int f(x)(Lh)^\mu(x)\partial_\mu\varphi(x)\,\mathrm{d}^4x = \langle b_f, \varphi\otimes h\rangle, \qquad W_g = \int g h_{00}^2,$$
$$\rho \in C_c^\infty(\mathbb{R}^4), \quad \rho \ge 0, \quad \rho\text{real, even and Euclidean-radial}, \quad \int\rho = 1,$$
$$S_\varepsilon = \rho_\varepsilon, \qquad \rho_\varepsilon(x) = \varepsilon^{-4}\rho(x/\varepsilon), \qquad B_{f,\varepsilon} = B_f[S_\varepsilon h, S_\varepsilon\varphi], \qquad b_{f,\varepsilon} = (\rho_\varepsilon\otimes\rho_\varepsilon)*b_f.$$

Even covariance (eq:Hhh, eq:Hscalar, eq:positive).

$$H_{hh\,\alpha\beta\rho\sigma} = P_{\alpha\beta\rho\sigma}\Delta_0^+, \qquad P_{\alpha\beta\rho\sigma} = \eta_{\alpha(\rho}\eta_{\sigma)\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta_{\rho\sigma},$$
$$H_{\varphi\varphi} = \Delta_m^+, \qquad H_{h\varphi} = H_{\varphi h} = 0, \qquad \Delta_m^+(x-y) = \int\frac{\mathrm{d}^3q}{(2\pi)^3\,2\omega_m(q)}e^{-i\omega_m(q)(t-s)+iq\cdot(\mathbf{x}-\mathbf{y})},$$
$$\omega_m(q) = \sqrt{|q|^2+m^2}, \qquad C_x^\mu H_{hh\,\rho\sigma} = \delta^\mu_{(\rho}\partial^x_{\sigma)}\Delta_0^+,$$
$$C_x^\mu C_y^\nu H_{hh} = -\tfrac{1}{2}\eta^{\mu\nu}\operatorname{square}_x\Delta_0^+ = 0, \qquad H_{ww} = 0.$$

Zero-Cauchy mode and mixed covariance (eq:K).

$$\mathrm{E}_0(e^{-iPt}) = -\frac{it}{2P}e^{-iPt} + \frac{i\sin(Pt)}{2P^2}, \qquad P = |p|,$$
$$K(x,y) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{ip\cdot(\mathbf{x}-\mathbf{y})}\Big[\Big(-\frac{t}{4P}+\frac{i}{8P^2}\Big)e^{-iP(t-s)} - \frac{i}{8P^2}e^{iP(t+s)}\Big].$$

The factors P^{-1} and P^{-2} are locally integrable in three momentum dimensions. No graviton positivity is assumed.

Profiles and spectral identity (eq:F, eq:spectral, lem:spectral).

$$\theta(u) = \widehat{\rho}(-u, u\mathbf{n}), \qquad M_\varepsilon(r) = \theta\big(\varepsilon\sqrt{r^2+m^2/2}\big),$$
$$K_\varepsilon = K\text{with the multiplier}\theta(\varepsilon P), \qquad F_{f,\varepsilon}(y,q) = \int f(x)K_\varepsilon(x,y)e^{-i\omega_m(q)t+iq\cdot\mathbf{x}}\,\mathrm{d}^4x,$$
$$\mathrm{ev}_0[\hbar^3](B_{f,\varepsilon}\star_H W_g\star_H B_{f',\varepsilon}) = \frac{1}{(2\pi)^3}\int g(y)\,\mathrm{d}^4y\int\omega_m(q)M_\varepsilon(|q|)^2F_{f,\varepsilon}(y,q)\overline{F_{f',\varepsilon}(y,q)}\,\mathrm{d}^3q \quad (f,f',g\text{real}).$$

There is one graph type. Its factor 2 counts the assignments of the two middle legs. The middle observable is not regularized.

Locked residue (eq:ray, eq:residue, thm:log).

$$A_f(y,\mathbf{n}) = \int_{\mathbb{R}} f(t,\mathbf{y}+(t+s)\mathbf{n})\,\mathrm{d}t, \qquad c_0 = \frac{1}{64(2\pi)^3},$$
$$\mathscr{C}(f,g,\dot{f}) = c_0 \int g(y)\,\mathrm{d}^4y \int_{S^2} A_f(y,\mathbf{n}) A_{\dot{f}}(y,\mathbf{n})\,\mathrm{d}\Omega(\mathbf{n}),$$
$$\mathscr{G}_\varepsilon(f,g,\dot{f}) = \mathscr{C}(f,g,\dot{f})\log(\varepsilon^{-1}) + O(1).$$

Here $\mathrm{d}\Omega$ is ordinary surface area measure: $\int_{S^2}\mathrm{d}\Omega = 4\pi$.

Off-shell identity (eq:reconstruction).

$$h = \mathrm{U}(\gamma_0 h, \gamma_1 h) + \mathrm{E}_0 j, \qquad j = \square h.$$

The source is retained. Equation does not set a classical field to zero.

endgroup

The exact distribution and intersection coordinates

Write

$$x = (t,\mathbf{x}), \qquad y = (s,\mathbf{y}), \qquad x' = (t',\mathbf{x}').$$

The scalar mass $m \geq 0$ and each admissible mollifier are fixed independently of ε.

Our Fourier convention is

$$\hat{v}(k) = \int e^{-ik\cdot z} v(z)\,\mathrm{d}z, \qquad v(z) = \int \frac{\mathrm{d}k}{(2\pi)^{\dim z}} e^{ik\cdot z}\hat{v}(k).$$

The dot product in this convention is the coordinate pairing, not the Lorentzian metric pairing. Radiality of ρ is Euclidean.

For a compact $K \subset X$, let

$$p_{K,N}(\Phi) = \max_{|\gamma|\leq N} \sup_X |\partial^\gamma \Phi|, \qquad \Phi \in \mathscr{D}_K(X).$$

Tests are extended by zero to the ambient Euclidean space when taking Fourier transforms. Their supports are separated from the boundary of X.

All estimates below can be taken with the single test order 64. Its size is inessential; fixing it makes independence

from the cutoff and the radial parameter explicit.

Strong convergence in $\mathscr{D}'(X)$ means uniform convergence on bounded subsets of $\mathscr{D}(X)$. A bounded subset of $\mathscr{D}(X)$ is contained and bounded in some fixed-support test space. Thus

$$|\langle U_\varepsilon - U, \Phi\rangle| \le C_K a(\varepsilon) p_{K,64}(\Phi), \qquad a(\varepsilon) \to 0,$$

on every compact implies strong convergence.

Positive-cutoff definition and regulator placement

Define

$$D_\varepsilon(z) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\omega_m(q)}{2} M_\varepsilon(|q|)^2 e^{-i\omega_m(q)z^0 + iq\cdot\mathbf{z}}.$$

Proposition (Exact positive-cutoff distribution and placement). *The field regularization and the mixed-kernel convolution in define the same insertion. Their numerical coefficient is*

$$T^\rho_\varepsilon(x,y,x') = 2K_\varepsilon(x,y)\overline{K_\varepsilon(x',y)}D_\varepsilon(x-x').$$

For every $\varepsilon > 0$, this is a smooth function on X, and

$$|\langle T^\rho_\varepsilon, \Phi\rangle| \le C_{K,\varepsilon,\rho,m} p_{K,0}(\Phi), \qquad \Phi \in \mathscr{D}_K(X).$$

Its pairing on real product tests is exactly .

Proof.

Field and kernel convolution. For check $\rho(z) = \rho(-z)$, transposition of convolution gives

$$\langle b_f, S_\varepsilon\varphi \otimes S_\varepsilon h\rangle = \langle (\check{\rho}_\varepsilon \otimes \check{\rho}_\varepsilon) * b_f, \varphi \otimes h\rangle.$$

Since ρ is even, this is the kernel convolution displayed in . The identity holds on arbitrary smooth configurations: the original mixed kernel has compact support, and the convolution kernels are compactly supported.

Plane-wave multiplier identities. The coefficients of C are constant, so

$$CS_\varepsilon h = S_\varepsilon C h$$

for every smooth h. For any constant tensor v,

$$S_\varepsilon(v e^{-iPz^0 + ip\cdot\mathbf{z}}) = \hat{\rho}(\varepsilon(-P,p)) v e^{-iPz^0 + ip\cdot\mathbf{z}} = \theta(\varepsilon P) v e^{-iPz^0 + ip\cdot\mathbf{z}}.$$

This applies in particular to the de Donder polarizations in . Applying the fixed operator $\mathrm{E}_0 C$ to this equality retains the scalar multiplier $\theta(\varepsilon P)$. This proves the displayed formula for K_ε, without commuting S_ε through E_0.

For the scalar leg, Euclidean radiality gives

$$\hat{\rho}(\varepsilon(-\omega_m(q), q)) = \theta\big(\varepsilon\sqrt{|q|^2 + m^2/2}\big) = M_\varepsilon(|q|).$$

The two scalar insertion slots therefore supply M_ε^2. The derivative factors are

$$(-i\omega_m(q))(+i\omega_m(q)) = \omega_m(q)^2;$$

multiplying the scalar spectral density $1/(2\omega_m(q))$ gives . The two insertion signs also multiply to +1. The middle observable is untouched.

Smoothness and the product-test identity. For fixed positive ε, every coordinate derivative of the momentum integrands in K_ε and D_ε is integrable, locally uniformly in the observations. At zero momentum the only graviton denominators are P^{-1} and P^{-2}, locally integrable in three dimensions. The scalar density is locally integrable also for $m = 0$. At infinity the profiles provide arbitrary polynomial decay.

Thus is a product of smooth functions. Its compact-test bound follows by a supremum on a compact neighborhood of K. Integrating against $f \otimes g \otimes \dot{f}$ and using yields .

Complex tests. The distribution is complex-linear in its joint test. On complex product tests, the spectral factors are

$$F_{f,\varepsilon}(y,q)\,\overline{F_{\bar{f},\varepsilon}(y,q)}.$$

The formula with $\overline{F_{\dot{f},\varepsilon}}$ alone applies to real $\dot{f}$. Kernel conjugation in does not conjugate the test.

Intersection coordinates

On $\mathbf{x} \neq \mathbf{y}$, define

$$a = t + s, \qquad b = t' + s, \qquad \mathbf{v} = \mathbf{x} - \mathbf{y}, \qquad \mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|}, \qquad \sigma = |\mathbf{v}| - a,$$
$$\mathbf{w} = \mathbf{x}' - \mathbf{y} - \frac{b}{a}(\mathbf{x} - \mathbf{y}), \qquad \mathbf{z} = \mathbf{y} + s\mathbf{n}, \qquad d = t - t'.$$

The chart domain has $a > 0$ and $a + \sigma > 0$. The inverse spatial formulas are

$$\mathbf{y} = \mathbf{z} - s\mathbf{n}, \qquad \mathbf{x} = \mathbf{y} + (a + \sigma)\mathbf{n}, \qquad \mathbf{x}' = \mathbf{y} + \frac{b}{a}(a + \sigma)\mathbf{n} + \mathbf{w}.$$

Consequently,

$$\mathrm{Rel} = \{\sigma = 0, \mathbf{w} = 0\}, \qquad \Delta_{13} = \{d = 0, \mathbf{w} = 0\}, \qquad \mathrm{Rel} \cap \Delta_{13} = \{\sigma = 0, \mathbf{w} = 0, d = 0\}.$$

For fixed times, the linear transformation to $(\mathbf{y}, \mathbf{v}, \mathbf{w})$ has determinant of absolute value one. Replacing $\mathbf{y}$ by $\mathbf{z} = \mathbf{y} + s\mathbf{n}$ is triangular with unit determinant. Polar coordinates in $\mathbf{v}$ therefore give

$$\mathrm{d}X = (a + \sigma)^2 \, \mathrm{d}^3\mathbf{z} \, \mathrm{d}\Omega(\mathbf{n}) \, \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}t' \, \mathrm{d}\sigma \, \mathrm{d}^3\mathbf{w}.$$

The full transformation, including the times, has the same Jacobian because its time block is the identity.

Replacing t' by $d = t - t'$ has absolute Jacobian one. There is no geometric $1/d$.

The locked residue

Let

$$Z = \mathbb{R}^3 \times S^2 \times (0, \infty)^3$$

with coordinates $(\mathbf{z}, \mathbf{n}, t, s, t')$, and set

$$\Psi(\mathbf{z}, \mathbf{n}, t, s, t') = ((t, \mathbf{z} + t\mathbf{n}), (s, \mathbf{z} - s\mathbf{n}), (t', \mathbf{z} + t'\mathbf{n})),$$

$$\mathrm{d}\nu = \mathrm{d}^3\mathbf{z} \, \mathrm{d}\Omega(\mathbf{n}) \, \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}t'.$$

Definition (Prescribed VII-C residue). The subtraction distribution is

$$r = c_0 \Psi_* v, \qquad c_0 = \frac{1}{64(2\pi)^3}.$$

Equivalently, for an arbitrary joint test,

$$\langle r, \Phi \rangle = c_0 \int \mathrm{d}s\ \mathrm{d}^3\mathbf{y}\ \mathrm{d}t\ \mathrm{d}t' \int_{S^2} \Phi(t, \mathbf{y} + (t+s)\mathbf{n}; ?s, \mathbf{y}; ?t', \mathbf{y} + (t'+s)\mathbf{n}) \mathrm{d}\Omega(\mathbf{n}).$$

The map Ψ is a proper embedding over compact subsets of X. Its inverse on its image is given by the observation times and

$$\mathbf{n} = \frac{\mathbf{x} - \mathbf{y}}{t + s}, \qquad \mathbf{z} = \mathbf{y} + s\mathbf{n}.$$

On the preimage of a compact in X, the time coordinates remain bounded and positive, the spatial coordinate $\mathbf{z}$ is bounded, and $\mathbf{n} \in S^2$. The preimage is closed in this bounded parameter region and hence compact.

It follows directly that r is a positive order-zero distribution, with support exactly Rel = $\Psi(Z)$. Its product-test pairing is . The separated-support nonzero witness is already established in VII-C, lem:separated.

Relative to the ambient density ,

$$r = c_0 a^{-2} \delta(\sigma) \delta^{(3)}(\mathbf{w}).$$

Indeed, multiplying this expression by $(a + \sigma)^2$ and integrating in $\sigma, \mathbf{w}$ gives exactly . There is no $\delta(d)$ in this measure.

If the same embedding and parameter density are written as $r_\Sigma = c_\Sigma \Psi_* v$, their normalization comparison is

$$r_\Sigma = 64(2\pi)^3 c_\Sigma\, r.$$

This paper always subtracts . The fact that the actual leading graph coefficient equals this prescribed distribution is proved in Proposition ; it is not built into the definition of the radial family.

Joint Fourier estimates and convergent sectors

The joint spatial transform

For fixed $(t, s, t', \mathbf{y})$, define

$$\mathscr{H}_{\Phi}(t, s, t', \mathbf{y}; \alpha, \beta) = \int_{\mathbb{R}^6} \Phi(t, \mathbf{x}; s, \mathbf{y}; t', \mathbf{x}') \times e^{i\alpha\cdot(\mathbf{x}-\mathbf{y}) - i\beta\cdot(\mathbf{x}'-\mathbf{y})}\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}'.$$

This preserves all correlations of the joint test.

We shall use the fixed momentum-decay exponent $N = 12$. Applying $(1-\Delta_{\mathbf{x}})^6(1-\Delta_{\mathbf{x}'})^6$ under the integral gives

$$|\partial_t^j \partial_{t'}^{j'} \mathscr{H}_{\Phi}| \le C_K p_{K,64}(\Phi)(1+|\alpha|)^{-12}(1+|\beta|)^{-12}, \qquad j + j' \le 8.$$

Only $24 + j + j' \le 32$ derivatives are needed for this estimate; order 64 also accommodates the fixed localizations used below. The variables $(t, s, t', \mathbf{y})$ range over a fixed compact projection of K.

Lemma (Shifted momentum bounds). *Let $r \ge 1$, $v \in S^2$, $\ell \in \{1, 2\}$, $k \in \{0, 1, 2\}$, and $N > k + 3$. Then*

$$\int_{\mathbb{R}^3} \frac{(1+|\alpha|)^{k-N}}{|\alpha - rv|^{\ell}}\, \mathrm{d}^3\alpha \le C r^{-\ell},$$

$$\int_{|\alpha| > r/2} \frac{(1+|\alpha|)^{k-N}}{|\alpha - rv|^{\ell}}\, \mathrm{d}^3\alpha \le C r^{3-\ell+k-N}.$$

The constants are uniform in r, v. For $0 \le r \le 1$, the first integral is uniformly bounded. In particular, all these assertions apply with $N = 12$.

Proof. On $|\alpha| \le r/2$, the denominator is at least $(r/2)^{\ell}$, and the remaining weight is integrable.

On $|\alpha| > r/2$, first consider $|\alpha - rv| \le r/2$. Here $|\alpha|$ is comparable to r, and

$$\int_{|p| \le r/2} |p|^{-\ell}\, \mathrm{d}^3 p = C_{\ell} r^{3-\ell}.$$

This part is bounded by $C r^{3-\ell+k-N}$. In the remaining part the denominator gives $C r^{-\ell}$, while

$$\int_{|\alpha| > r/2} (1+|\alpha|)^{k-N}\, \mathrm{d}^3\alpha \le C r^{3+k-N}.$$

This proves the tail bound. Since $N > k + 3$, the tail also satisfies the full-integral bound.

For $0 \le r \le 1$, split into a fixed-size neighborhood of $\alpha = rv$ and its complement. Local integrability of $|p|^{-\ell}$ treats the first region, and the fixed polynomial weight treats the second.

Certificate. These bounds include all momentum ratios and all directions. They use no time separation, regulator decay, or angular stationary-phase hypothesis.

Exact branch integrals

Set

$$a_D(t,P) = -\frac{t}{4P} + \frac{i}{8P^2}, \qquad a_R(t,P) = -\frac{i}{8P^2}, \qquad \varepsilon_D = -1, \quad \varepsilon_R = 1.$$

Throughout the momentum calculation we use the single substitution

$$p = \alpha - q, \qquad p' = \beta - q, \qquad P = |\alpha - q|, \qquad P' = |\beta - q|.$$

For $A, B \in \{D, R\}$, define

$$Q_\varepsilon^{AB}(q;\Phi) = \int \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}t'\ \mathrm{d}^3\mathbf{y} \int \frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6} \theta(\varepsilon P)\theta(\varepsilon P') \times a_A(t,P)\overline{a_B(t',P')}\, e^{i[(\varepsilon_A P - \omega_m(q))t - (\varepsilon_B P' - \omega_m(q))t' + (P-P')s]}\, \mathscr{H}_\Phi(t,s,t',\mathbf{y};\alpha,\beta).$$

For $\varepsilon = 0$, the two profile factors are replaced by one. For each fixed q, the integral is absolutely convergent after taking the joint spatial transform.

The exact positive-cutoff identity is

$$\langle T_\varepsilon^\rho, \Phi\rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \omega_m(q) M_\varepsilon(|q|)^2 \sum_{A,B \in \{D,R\}} Q_\varepsilon^{AB}(q;\Phi)\, \mathrm{d}^3 q.$$

The spatial signs follow from

$$p \cdot (\mathbf{x} - \mathbf{y}) - p' \cdot (\mathbf{x}' - \mathbf{y}) + q \cdot (\mathbf{x} - \mathbf{x}') = \alpha \cdot (\mathbf{x} - \mathbf{y}) - \beta \cdot (\mathbf{x}' - \mathbf{y}).$$

Define S_ε^ρ by taking all four branches in on $|q| \le 1$, and only DD, DR, RD on $|q| > 1$. Define S by the same prescription with all multipliers replaced by one. The high-$|q|$ integrals defining S are understood after performing the compact time integrations in .

Lemma (Low-frequency and direct sectors). *The distribution S exists, is independent of ρ, and satisfies*

$$|\langle S_\varepsilon^\rho - S, \Phi\rangle| \le C_{K,\rho,m}\,\varepsilon\, p_{K,64}(\Phi), \qquad 0 < \varepsilon \le \tfrac{1}{2}.$$

Proof. The profiles are bounded and globally Lipschitz, with value one at zero. Telescoping the product of multipliers gives

$$\left| M_\varepsilon(|q|)^2 \theta(\varepsilon P)\theta(\varepsilon P') - 1 \right| \le C_{\rho,m}\varepsilon(1 + |q| + P + P').$$

For $|q| \le 1$, and the bounded-r part of Lemma give an integrable bound. The extra powers in are harmless: $P \le |\alpha| + 1$ and $P' \le |\beta| + 1$. Thus this sector has a cutoff-free limit with $O(\varepsilon)$ error.

For $r = |q| \ge 1$, suppose the left branch is direct. Its t-frequency is $-(P + \omega_m(q))$. Integrating eight times by parts in t gains

$$(P + \omega_m(q))^{-8} \le r^{-8}.$$

The amplitude is at most linear in t; its differentiated coefficients are still bounded by $C_K(P^{-1} + P^{-2})$. The other amplitude is bounded by $C_K(P'\wedge{-}1 + P'\wedge{-}2)$. If only the right branch is direct, use t' instead.

The shifted bounds show that the two momentum integrations contribute at most Cr^{-2}. Since $r^2\omega_m(r) \le C_m r^3$, the unregulated radial majorant is Cr^{-7}. For the regulated difference,

$$P \le r + |\alpha|, \qquad P' \le r + |\beta|,$$

and costs at most $C\varepsilon r$ after the weighted integrations. The resulting radial majorant is $C\varepsilon r^{-6}$. Both are integrable.

All integrations by parts are performed against the full compact joint test. No restriction to an observation diagonal is used.

Certificate. This lemma holds on every compact in X, including ordinary diagonals and the reflected intersection. It covers bounded scalar momentum and every sector with a direct branch. The test order is 64, independent of ε; the error is $O_{K,\rho,m}(\varepsilon)$, without a lower bound on $|t - t'|$.

The reflected radial family: normalization and Lemma A

For $q = rv$, $r \geq 1$, set

$$P = |\alpha - rv|, \qquad P' = |\beta - rv|, \qquad \omega(r) = \sqrt{r^2 + m^2},$$

in accordance with .

The radial families are defined intrinsically from the exact branch integrals:

$$\langle A(r), \Phi \rangle = \frac{r^3 \omega(r)}{(2\pi)^3} \int_{S^2} Q_0^{RR}(rv; \Phi) \mathrm{d}\Omega(v),$$

$$\langle A_\varepsilon^\rho(r), \Phi \rangle = \frac{r^3 \omega(r)}{(2\pi)^3} \int_{S^2} Q_\varepsilon^{RR}(rv; \Phi) \mathrm{d}\Omega(v).$$

No residue constant has been inserted in these definitions. They give the exact decomposition

$$T_\varepsilon^\rho = S_\varepsilon^\rho + \int_1^\infty M_\varepsilon(r)^2 A_\varepsilon^\rho(r) \frac{\mathrm{d}r}{r}.$$

The use of r as the radial integration variable is distinct from its use for the fixed residue distribution. In expressions such as $A(r) - r$, the second term is the distribution .

Normalization ledger

Proposition (Normalization ledger). *The leading reflected–reflected radial coefficient derived from equals the prescribed distribution on all joint tests. In particular, its constant is* $c_0 = 1/(64(2\pi)^3)$*, with the parameter density in .*

Proof. The calculation is recorded in six identities.

L1: exact pointwise kernel. Proposition gives

$$T_\varepsilon^\rho = 2K_\varepsilon(x, y)\overline{K_\varepsilon(x', y)}D_\varepsilon(x - x').$$

L2: reflected amplitudes and conjugation. The reflected terms of the two mixed covariances have amplitudes

$$a_R(t,P) = -\frac{i}{8P^2}, \qquad \overline{a_R(t',P')} = \frac{i}{8P'^2}, \qquad a_R(t,P)\overline{a_R(t',P')} = \frac{1}{64P^2P'^2}.$$

The corresponding time phases are, separately, $e^{iP(t+s)}$ and $e^{-iP'(t'+s)}$. The right spatial phase is $e^{-ip'\cdot(\mathbf{x}'-\mathbf{y})}$. These are conjugations of the kernel, not of Φ.

L3: scalar density. The scalar factor is

$$D_\varepsilon(z) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\omega_m(q)}{2} M_\varepsilon(|q|)^2 e^{-i\omega_m(q)z^0 + iq\cdot\mathbf{z}}.$$

L4: combined numerical prefactor. Consequently the reflected–reflected integrand has prefactor

$$2\cdot\frac{1}{64P^2P'^2}\cdot\frac{\omega_m(q)}{2} = \frac{\omega_m(q)}{64P^2P'^2}, \qquad \text{with measure} \frac{\mathrm{d}^3q}{(2\pi)^3}.$$

The assignment factor in L1 has now been exhausted. No further factor of two occurs in $A(r)$.

L5: polar measure and radial normalization. With ordinary sphere area measure,

$$\mathrm{d}^3q = r^2\,\mathrm{d}r\,\mathrm{d}\Omega(v), \qquad \frac{r^2\omega(r)}{P^2P'^2}\mathrm{d}r = \frac{r^3\omega(r)}{P^2P'^2}\frac{\mathrm{d}r}{r}.$$

This explains the extra r^3 in , rather than adding a normalization by hand.

L6: translation and Fourier inversion. For fixed q, is a translation and has Jacobian one. The two mixed-covariance measures become exactly $\mathrm{d}^3\alpha\ \mathrm{d}^3\beta/(2\pi)^6$. The Fourier inversion identity is

$$\left|\det\frac{\partial(p,p')}{\partial(\alpha,\beta)}\right| = 1, \qquad \frac{\mathrm{d}^3p\ \mathrm{d}^3p'}{(2\pi)^6} = \frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6},$$
$$\int \frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6} e^{-ia\nu\cdot\alpha + ib\nu\cdot\beta}\mathscr{H}_\Phi(t,s,t',\mathbf{y};\alpha,\beta)$$
$$= \Phi(t,\mathbf{y}+a\nu;?s,\mathbf{y};?t',\mathbf{y}+b\nu).$$

In particular, the two factors $(2\pi)^{-3}$ supplied by the mixed covariances are the normalization of the two Fourier inversions. They cancel the factors $(2\pi)^3$ from the unnormalized Fourier delta identities; they are not canceled a second time against another Fourier measure.

For the leading coefficient, replace the amplitude factors P^{-2}, P'^{-2} by r^{-2}, r^{-2} and $\omega(r)$ by r. In the phase, however, retain the first-order expansions

$$P = r - v \cdot \alpha, \qquad P' = r - v \cdot \beta, \qquad \omega(r) = r.$$

Substituting $P = P' = r$ into the phase itself would be incorrect. The retained time phase is exactly the inversion phase in L6.

The only remaining numerical factor is therefore

$$\frac{1}{(2\pi)^3}\frac{1}{64} = c_0.$$

After L6, the remaining measure is $\mathrm{d}t\ \mathrm{d}s\ \mathrm{d}t'\ \mathrm{d}^3\mathbf{y}\,\mathrm{d}\Omega(v)$. Changing $\mathbf{y}$ to $\mathbf{z} = \mathbf{y} + sv$ has unit Jacobian, so this is precisely the pushforward density . We have proved

$$\text{boxed}\begin{array}{l} A_{\mathrm{lead}} = r \quad \text{in}\,\mathscr{D}'(X), \qquad c_0 = \frac{1}{64(2\pi)^3}, \\ \text{leading RR contribution} = \langle r, \Phi\rangle \frac{\mathrm{d}r}{r}. \end{array}$$

The cutoff factors $M_\varepsilon(r)^2\theta(\varepsilon P)\theta(\varepsilon P')$ have not been used in this leading identification. They are treated by the replacement estimate below. Direct-branch amplitudes do not enter this calculation.

As a consequence of the ledger, the intrinsic definition can now be written

$$\langle A(r), \Phi\rangle = c_0 r^3 \omega(r) \int_{S^2} \mathrm{d}\Omega(v) \int \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}t'\ \mathrm{d}^3\mathbf{y} \times \int \frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6 P^2 P'^2} e^{i[(P-\omega(r))t - (P'-\omega(r))t' + (P-P')s]} \mathscr{H}_\Phi(t, s, t', \mathbf{y}; \alpha, \beta).$$

The formula for $A^\rho_\varepsilon(r)$ has the two additional factors $\theta(\varepsilon P)\theta(\varepsilon P')$.

Lemma A: exact phase and complete error budget

Lemma (Joint-test radial asymptotic: Lemma A). *For every compact $K \subset X$,*

$$|\langle A(r) - r, \Phi\rangle| \le C_{K,m} r^{-1} p_{K,64}(\Phi), \qquad r \ge 1.$$

In particular,

$$|\langle A(r), \Phi\rangle| \le C_{K,m} p_{K,64}(\Phi).$$

The estimate holds without a positive lower bound on $|t - t'|$.

Proof. We retain the exact phase, form an exact difference, and only then estimate its terms.

1. Exact phase. Set

$$\delta_r^{\nu}(\alpha) = |\alpha - r\nu| - r + \nu \cdot \alpha, \qquad \varphi_0 = -a\nu \cdot \alpha + b\nu \cdot \beta.$$

The temporal phase in is exactly

$$\varphi_0 + a\delta_r^{\nu}(\alpha) - b\delta_r^{\nu}(\beta) + (r - \omega(r))d.$$

Before the spatial integration defining $\mathscr{H}_{\Phi}$, the exact full phase in the incidence chart is

$$\begin{aligned}\Theta_r = \quad & (a\alpha - b\beta) \cdot (\mathbf{n} - \nu) + \sigma\mathbf{n} \cdot \left(\alpha - \tfrac{b}{a}\beta\right) - \beta \cdot \mathbf{w} \\ & + a\delta_r^{\nu}(\alpha) - b\delta_r^{\nu}(\beta) + (r - \omega(r))d.\end{aligned}$$

Thus both angular variables and all of $(\sigma, \mathbf{w}, d)$ are retained. The phase φ_0 is the leading *temporal* phase in Cartesian Fourier space; the spatial phase is already inside $\mathscr{H}_{\Phi}$. One does not delete the first line of in an incidence-coordinate integral.

2. Exact central–tail decomposition. Let

$$\mathscr{C}_r = \{|\alpha| \le r/2,\ |\beta| \le r/2\}, \qquad \lambda_r = \frac{\omega(r)}{r}, \qquad J_r = \frac{r^4}{P^2 P'\wedge 2},$$

and abbreviate, within this proof,

$$u_{\delta} = a\delta_r^{\nu}(\alpha) - b\delta_r^{\nu}(\beta), \qquad u_{\omega} = (r - \omega(r))d.$$

All integrals over $\mathscr{C}_r$ or its complement below include the remaining variables, with measure

$$\mathrm{d}\Omega(\nu)\ \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}t'\ \mathrm{d}^3\mathbf{y}\,\frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6}.$$

Write $\mathscr{H}_{\Phi}$ at its full arguments from . Define

$$I_{\mathrm{cent}}^{\mathrm{lead}} = c_0 \int_{\mathscr{C}_r} e^{i\varphi_0}\mathscr{H}_{\Phi},$$

$$E_P = c_0 \int_{\mathscr{C}_r} e^{i\varphi_0}\mathscr{H}_{\Phi}\lambda_r(J_r - 1)e^{i(u_{\delta} + u_{\omega})},$$

$$E_\delta = c_0 \int_{\mathscr{C}_r} e^{i\varphi_0} \mathscr{H}_\Phi \lambda_r (e^{iu_\delta} - 1) e^{iu_\omega},$$

$$E_\omega = c_0 \int_{\mathscr{C}_r} e^{i\varphi_0} \mathscr{H}_\Phi (\lambda_r e^{iu_\omega} - 1),$$

$$T_{\text{tail}} = c_0 \int_{\mathscr{C}_r^{\varepsilon}} e^{i\varphi_0} \mathscr{H}_\Phi \lambda_r J_r e^{i(u_\delta + u_\omega)}, \qquad R_{\text{tail}}^{\text{lead}} = c_0 \int_{\mathscr{C}_r^{\varepsilon}} e^{i\varphi_0} \mathscr{H}_\Phi .$$

These definitions precede all absolute-value estimates. The algebraic identity behind them is

$$\lambda_r J_r e^{i(u_\delta + u_\omega)} - 1 = \lambda_r (J_r - 1) e^{i(u_\delta + u_\omega)} + \lambda_r (e^{iu_\delta} - 1) e^{iu_\omega} + (\lambda_r e^{iu_\omega} - 1).$$

Consequently,

$$\langle A(r), \Phi \rangle = I_{\text{cent}}^{\text{lead}} + E_P + E_\delta + E_\omega + T_{\text{tail}},$$

$$I_{\text{cent}}^{\text{lead}} = \langle r, \Phi \rangle - R_{\text{tail}}^{\text{lead}}.$$

The second equality is the full Fourier inversion proved in the ledger, minus its leading tail. In particular, the central leading term is not counted again after subtracting r. With

$$E_{\text{tail}} = T_{\text{tail}} - R_{\text{tail}}^{\text{lead}},$$

the exact difference is

$$\langle A(r) - r, \Phi \rangle = E_P + E_\delta + E_\omega + E_{\text{tail}}.$$

3. Amplitude error E_P. On $\mathscr{C}_r$,

$$\frac{r}{2} \le P, P' \le \frac{3r}{2}, \qquad \left| \frac{r^2}{P^2} - 1 \right| = \frac{|r - P|(r + P)}{P^2} \le C \frac{|\alpha|}{r}.$$

The corresponding bound holds for P'. Since each ratio r^2/P^2 is bounded by 4,

$$|J_r - 1| \le C \frac{|\alpha| + |\beta|}{r}, \qquad J_r \le 16.$$

Also $\lambda_r \le \sqrt{1 + m^2}$ for $r \ge 1$. Using in gives

$$|E_P| \leq \frac{C_{K,m}}{r} p_{K,64}(\Phi) \int_{\mathbb{R}^6} \frac{|\alpha| + |\beta|}{(1+|\alpha|)^{12}(1+|\beta|)^{12}} \, \mathrm{d}^3\alpha \, \mathrm{d}^3\beta \leq \frac{C'_{K,m}}{r} p_{K,64}(\Phi).$$

4. Quadratic phase error E_{δ}. For $|\alpha| \leq r/2$, the exact identity

$$\delta_r^v(\alpha) = \frac{|\alpha|^2 - (v \cdot \alpha)^2}{|\alpha - rv| + r - v \cdot \alpha} \quad \text{implies} \quad |\delta_r^v(\alpha)| \leq \frac{|\alpha|^2}{r}.$$

The denominator is at least r in this region. On a compact observation set, a, b have fixed upper bounds. Therefore

$$|u_\delta| \leq \frac{C_K}{r}(|\alpha|^2 + |\beta|^2).$$

Applying $|e^{iu} - 1| \leq |u|$ to the exact difference in , and then , yields

$$|E_\delta| \leq \frac{C_{K,m}}{r} p_{K,64}(\Phi) \int_{\mathbb{R}^6} \frac{|\alpha|^2 + |\beta|^2}{(1+|\alpha|)^{12}(1+|\beta|)^{12}} \, \mathrm{d}^3\alpha \, \mathrm{d}^3\beta \leq \frac{C'_{K,m}}{r} p_{K,64}(\Phi).$$

No expansion of δ_r^v is made in the tail.

5. Mass and time-difference error E_{ω}. The two relevant bounds are

$$\omega(r) - r = \frac{m^2}{\omega(r) + r} \leq \frac{m^2}{2r}, \qquad \lambda_r - 1 = \frac{m^2}{r(\omega(r) + r)} \leq \frac{m^2}{2r^2}.$$

The exact factor in splits as

$$\lambda_r e^{iu_\omega} - 1 = (\lambda_r - 1)e^{iu_\omega} + (e^{iu_\omega} - 1).$$

In particular,

$$|u_\omega| = |(r - \omega(r))d| \leq \frac{m^2|d|}{2r}.$$

Only an upper bound on $|d|$ over the compact observation projection is used. At $d = 0$, this phase error vanishes exactly. Equations and – give

$$|E_\omega| \leq C_{K,m}(r^{-1} + r^{-2}) p_{K,64}(\Phi).$$

This is the required estimate through Δ_{13}; it does not use division by d.

6. Exact tail E?$_{\text{ai}}$?. Let $N = 12$. The union bound for $\mathscr{C}_r^{\varepsilon}$, together with – for $\ell = 2, k = 0$, gives the complete majorant

$$|T_{\text{tail}}| \le C_{K,m} p_{K,64}(\Phi)\,\underbrace{r^4}_{\text{prefactor}}\Big(\underbrace{r^{1-N}}_{\alpha\text{-tail}}\underbrace{r^{-2}}_{\text{full}\beta} + \underbrace{r^{-2}}_{\text{full}\alpha}\underbrace{r^{1-N}}_{\beta\text{-tail}}\Big)$$

$$\le C'_{K,m} r^{3-N} p_{K,64}(\Phi) = C'_{K,m} r^{-9} p_{K,64}(\Phi).$$

Here the prefactor bound is $r^3\omega(r) \le C_m r^4$. The shifted-tail exponent is $3 - 2 + 0 - N = 1 - N$, whereas the full shifted integral is bounded by Cr^{-2}.

This estimate includes $P \approx 0$ and $P' \approx 0$, which occur in the shifted tails. Those singularities are treated by local integrability, not by a phase expansion.

For the leading tail there are no shifted denominators. Directly,

$$\int_{|\alpha| > r/2} (1 + |\alpha|)^{-N} \mathrm{d}^3\alpha \le C r^{3-N},$$

and similarly for β. Hence

$$|R^{\text{lead}}_{\text{tail}}| + |E_{\text{tail}}| \le C_{K,m} r^{3-N} p_{K,64}(\Phi) = C_{K,m} r^{-9} p_{K,64}(\Phi).$$

7. Coordinate change: E_chart = 0. The estimate has been proved in Cartesian Fourier space. On a fixed incidence-coordinate patch, rewriting the same distribution with introduces exactly the density and the phase . It introduces no additional error:

$$E_{\text{chart}} = 0.$$

For a direct justification, first insert compact smooth cutoffs in α, β. The remaining momentum singularities $P^{-2}, P'\wedge{-2}$ are locally integrable, so the ordinary coordinate change is legitimate. Remove those auxiliary cutoffs after the spatial integration defining $\mathscr{H}_{\Phi}$; and Lemma justify that limit. This identifies the coordinate expression with the original distribution, without restricting it to a diagonal. The Jacobian has no $1/d$.

8. Sum of the errors. The complete budget is

$$\begin{array}{cc} c|c\text{term} & \text{bound divided by} p_{K,64}(\Phi) \\ \text{hline}\, E_P & C_{K,m} r^{-1} \\ E_\delta & C_{K,m} r^{-1} \\ E_\omega & C_{K,m}(r^{-1} + r^{-2}) \\ E_{\text{tail}} & C_{K,m} r^{-9} \\ E_{\text{chart}} & 0 \end{array}$$

Summing in proves . The uniform bound on $A(r)$ follows because r is an order-zero distribution.

Certificate. Lemma A holds on every compact of X, with test order 64, independently of the regulator. The proof covers the comparable central sector and all noncomparable shifted tails, uniformly in the momentum direction. It uses only upper bounds on $a, b, |d|$. Thus it applies unchanged on neighborhoods of Rel $\cap\, \Delta_{13}$.

The estimate is an equality and bound on all joint tests. In particular, an additional order-r^0 contribution proportional to $\delta(d)$, a normal derivative of the residue, or any other nonzero distribution could not be hidden in its error: it would contradict on a test detecting that distribution.

Replacement and the complete ultraviolet split

Replacement of the two graviton multipliers

Lemma (Insertion-regulator replacement). *For every compact $K \subset X$,*

$$\left|\langle A_\varepsilon^\rho(r) - \theta(\varepsilon r)^2 A(r), \Phi\rangle\right| \le C_{K,\rho,m}\, \varepsilon\, p_{K,64}(\Phi), \qquad r \ge 1, \quad 0 < \varepsilon \le 1.$$

Only boundedness and a global Lipschitz bound for θ are needed for this estimate.

Proof.

1. Exact multiplier difference. Before taking absolute values, use the identity

$$\begin{aligned} &\theta(\varepsilon P)\theta(\varepsilon P') - \theta(\varepsilon r)^2 \\ &= \theta(\varepsilon P')(\theta(\varepsilon P) - \theta(\varepsilon r)) + \theta(\varepsilon r)(\theta(\varepsilon P') - \theta(\varepsilon r)). \end{aligned}$$

Each difference has the exact integral remainder

$$\theta(\varepsilon P)-\theta(\varepsilon r)=\varepsilon(P-r)\int_0^1 \theta'(\varepsilon(r+\tau(P-r)))\,\mathrm{d}\tau,$$

and likewise with P'.

Using the phase notation from the proof of Lemma A, inserting the left-hand side of into gives exactly

$$\begin{aligned}&\langle A_\varepsilon^{\rho}(r)-\theta(\varepsilon r)^2 A(r),\Phi\rangle\\ &=c_0 r^3\omega(r)\int \frac{\theta(\varepsilon P)\theta(\varepsilon P')-\theta(\varepsilon r)^2}{P^2P'\hat{}2}e^{i[\varphi_0+u_\delta+u_\omega]}\mathscr{H}_\Phi\\ &\qquad\times \mathrm{d}\Omega(v)\ \mathrm{d}t\ \mathrm{d}s\ \mathrm{d}t'\ \mathrm{d}^3\mathbf{y}\,\frac{\mathrm{d}^3\alpha\ \mathrm{d}^3\beta}{(2\pi)^6}.\end{aligned}$$

Here the phase is the same exact phase used in Lemma A. No approximation of it is made.

Since

$$|P-r|\le|\alpha|,\qquad |P'-r|\le|\beta|,$$

equations – imply

$$\left|\theta(\varepsilon P)\theta(\varepsilon P')-\theta(\varepsilon r)^2\right|\le\|\theta\|_\infty\|\theta'\|_\infty\varepsilon(|\alpha|+|\beta|).$$

This uses no decay when P or P' is small.

2. Central region. On the region $\mathscr{C}_r$ of Lemma A, $r^3\omega(r)/(P^2P'\hat{}2)\le C_m$. Taking absolute values in the already formed difference gives

$$\begin{aligned}|\text{central part of eq:replacementexact}|\ &\le C_{K,\rho,m}\varepsilon\,p_{K,64}(\Phi)\int_{\mathbb{R}^6}\frac{|\alpha|+|\beta|}{(1+|\alpha|)^{12}(1+|\beta|)^{12}}\,\mathrm{d}^3\alpha\ \mathrm{d}^3\beta\\ &\le C'_{K,\rho,m}\varepsilon\,p_{K,64}(\Phi).\end{aligned}$$

3. Shifted tails, including P ≈ 0. Use the same complement $\mathscr{C}_r^{c}$ as in Lemma A, and retain $N=12$. For the α-tail, apply to the α-integral and to the full β-integral. The extra momentum factor is absorbed using $|\alpha|\le 1+|\alpha|$ or $|\beta|\le 1+|\beta|$. The following table records the parameters of those two estimates. Every row includes the prefactor $r^3\omega(r)\le C_m r^4$; the common factor $C_{K,\rho,m}p_{K,64}(\Phi)$ is suppressed.

1.25 toprule Extra factor α-tail: (k,ℓ) and bound Full β: (k,ℓ) and bound Radial majorant

midrule $\varepsilon|\alpha|\ (1,2) : Cr^{2-N}\ (0,2) : Cr^{-2}\ C\varepsilon r^{4-N}$

$\varepsilon|\beta|\ (0,2) : Cr^{1-N}\ (1,2) : Cr^{-2}\ C\varepsilon r^{3-N}$

bottomrule

Indeed, the tail powers are $3-2+1-N = 2-N$ and $3-2+0-N = 1-N$. The full-integral estimate with $k = 1, \ell = 2$ still gives Cr^{-2}, since $N = 12 > 1 + 3$; no additional power of r is lost there. The β-tail has the same two bounds after interchanging α and β. The union bound therefore yields

$$\begin{aligned} |\text{tail part of eq:replacementexact}| &\le C_{K,\rho,m}\varepsilon(r^{4-N} + r^{3-N})p_{K,64}(\Phi) \\ &= C_{K,\rho,m}\varepsilon(r^{-8} + r^{-9})p_{K,64}(\Phi) \\ &\le C'_{K,\rho,m}\varepsilon\, p_{K,64}(\Phi). \end{aligned}$$

The shifted-tail bounds include the locally integrable singularities at $P = 0$ and $P' = 0$. Combining and proves the claim.

Certificate. The replacement estimate holds on all of K, including Rel $\cap\ \Delta_{13}$, with order 64, uniformly in r. Its central error is $O(\varepsilon)$, and its tail is $O(\varepsilon(r^{-8} + r^{-9}))$. No small-argument decay of a graviton multiplier is used. The estimate does not replace any multiplier in the direct-sector distribution S^{ρ}_{ε}, and it does not yet replace the scalar multiplier M_{ε}.

The ultraviolet decomposition

Proposition (Regulator-independent ultraviolet decomposition). *The exact family has the decomposition*

$$\text{boxed}\ T^{\rho}_{\varepsilon} = S + \int_{1}^{\infty} \theta(\varepsilon r)^4 A(r)\frac{\mathrm{d}r}{r} + E^{\rho}_{\varepsilon},$$

where $S, A(r)$ are independent of ρ, and

$$|\langle E^{\rho}_{\varepsilon}, \Phi\rangle| \le C_{K,\rho,m}\varepsilon(1 + |\log\varepsilon|)p_{K,64}(\Phi), \qquad 0 < \varepsilon \le \tfrac{1}{2}.$$

Proof. The ledger identifies the leading coefficient with r, and Lemma A gives an integrable radial remainder $A(r) - r$. For the finite-cutoff identity, subtract the radial term in from . The exact error is the sum of $S^{\rho}_{\varepsilon} - S$ and

$$\int_1^\infty M_\varepsilon(r)^2 (A_\varepsilon^\rho(r) - \theta(\varepsilon r)^2 A(r)) \frac{\mathrm{d}r}{r},$$

$$\int_1^\infty \theta(\varepsilon r)^2 (M_\varepsilon(r)^2 - \theta(\varepsilon r)^2) A(r) \frac{\mathrm{d}r}{r}.$$

The first difference is $O(\varepsilon)$ by Lemma . For , Lemma and

$$|M_\varepsilon(r)| \le C_{\rho,N} (1 + \varepsilon r)^{-N}$$

give the bound

$$C_{K,\rho,m} \varepsilon p_{K,64}(\Phi) \int_1^\infty (1 + \varepsilon r)^{-N} \frac{\mathrm{d}r}{r} \le C'_{K,\rho,m} \varepsilon (1 + |\log \varepsilon|) p_{K,64}(\Phi).$$

It is the scalar multiplier that supplies this large-r decay.

The scalar replacement is a separate identity and estimate:

$$\begin{aligned} \sqrt{r^2 + m^2/2} - r &= \frac{m^2/2}{\sqrt{r^2 + m^2/2} + r} \le \frac{m^2}{4r}, \\ |M_\varepsilon(r) - \theta(\varepsilon r)| &\le \frac{m^2}{4} \|\theta'\|_\infty \varepsilon r^{-1}. \end{aligned}$$

The boundedness of the profiles and of $A(r)$ therefore makes $O(\varepsilon)$, since $\int_1^\infty r^{-2}\,\mathrm{d}r < \infty$. All estimates use the same compact-test seminorm.

Intersection and sector audit

The full intersection pairing

For a test supported in an incidence-coordinate patch, the exact pairing corresponding to is

$$\begin{aligned} \langle A(r), \Phi \rangle = \quad & c_0 r^3 \omega(r)\, \mathrm{Os} \int \frac{(a+\sigma)^2 \widehat{\Phi} e^{i\Theta_r}}{P^2 P' \wedge 2} \\ & \times \mathrm{d}^3\mathbf{z}\, \mathrm{d}\Omega(\mathbf{n})\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}d\, \mathrm{d}\sigma\, \mathrm{d}^3\mathbf{w}\, \mathrm{d}\Omega(v)\, \frac{\mathrm{d}^3\alpha\, \mathrm{d}^3\beta}{(2\pi)^6}, \end{aligned}$$

where $t' = t - d > 0$ and Θ_r is . The oscillatory notation means the Cartesian-Fourier distribution rewritten by the cutoff-and-coordinate procedure in Lemma A. It is not a new product of singular distributions.

The leading Fourier inversion imposes

$$\mathbf{x} - \mathbf{y} = av, \qquad \mathbf{x}' - \mathbf{y} = bv.$$

Only after that inversion may one conclude

$$v = \mathbf{n}, \qquad \sigma = 0, \qquad \mathbf{w} = 0.$$

The positive radius $a = t + s$, not $a - b = d$, determines the first angular constraint. Neither this identification nor its density requires division by d.

Normal jets as a consequence, not a second proof

On a fixed patch, write $v = (\sigma, \mathbf{w})$, and let u denote all tangential variables, including d. A fixed local cutoff and the integral Taylor formula give

$$\widetilde{\Phi}(u, v) = \chi(v)\widetilde{\Phi}(u, 0) + \sum_{j=1}^{4} v_j \Phi_j(u, v),$$

with smooth compactly supported Φ_j and finite-order seminorm bounds on a slightly larger patch. The residue kills the normal-vanishing terms. Lemma A already controls those terms and the trace term uniformly through $d = 0$. This observation does not restrict $A(r)$ to the relation and is not used as an alternative proof of the radial estimate.

Observation coverage

The estimates are global in the Cartesian observation variables, so they cover the following regions.

1. Off Rel, the residue vanishes on supported tests. Equations and then give convergence without a logarithmic subtraction.

2. Near Rel $\setminus \Delta_{13}$, the same estimates apply without using a lower bound on $|d|$.

3. Near Rel $\cap \Delta_{13}$, the exact phase and density are and . The mass-phase bound is uniform through $d = 0$.

4. The remaining ordinary diagonals and the region $\mathbf{x} = \mathbf{y}$, where the incidence chart is unavailable, are covered by the Cartesian Fourier argument. Moreover, Rel $\cap \Delta_{12}$ = Rel $\cap \Delta_{23} = \varnothing$ in X, because the corresponding reflected

spatial distances are $t + s > 0$ and $t^{'} + s > 0$.

These are assertions on joint tests supported in open observation neighborhoods, not tests confined to positive-codimension strata. A finite smooth partition on any compact gives the corresponding regional estimates with the same test order.

Integration-sector coverage

1.14 toprule Sector Controlling calculation

midrule All frequencies large and comparable The normalization ledger and the central errors $E_{P}, E_{\delta}, E_{\omega}$.

Only a proper subset large Bounded q belongs to Lemma . For large q, a small graviton momentum forces a shifted tail, covered by .

Ratios tending to zero or infinity The shifted-tail bounds include both limits and $P = 0, P^{'} = 0$, with their locally integrable denominators.

A direct branch present Time integration by parts in a frequency $P + \omega(r)$ or $P^{'} + \omega(r)$, independently of d.

Angular degeneracy Uniform estimates on S^{2}, not angular stationary phase. The two directions in Θ_{r} are kept distinct.

A graviton cutoff argument tends to zero The exact telescoping and mean-value identities –; only boundedness and Lipschitz continuity are used.

Bounded frequencies The common distribution S, with $O(\varepsilon)$ convergence.

bottomrule

The partitions at $|q| = 1$ and $|\alpha|, |\beta| = r/2$ are independent of ε. No derivative of a cutoff-dependent partition is omitted.

Scope audit. This numerical subtraction leaves the VII-C coefficient-continuity obstruction under its stated normal-topology hypotheses unchanged [1, thm:coefficient].

Strong finite part and dyadic certificates

Define the real profile constant

$$\kappa_\rho = \int_0^1 \frac{\theta(u)^4 - 1}{u}\,\mathrm{d}u + \int_1^\infty \frac{\theta(u)^4}{u}\,\mathrm{d}u.$$

The profile is real, even, Schwartz, and equals one at zero. Hence $\theta(u)^4 - 1 = O(u^2)$, and both integrals converge.

Define the common distribution

$$U_m = S + \int_1^\infty (A(r) - r)\frac{\mathrm{d}r}{r}.$$

This integral is defined by its joint-test pairings. They are measurable in r, and Lemma A gives the integrable bound

$$|\langle A(r) - r, \Phi\rangle|\frac{1}{r} \le C_{K,m} r^{-2} p_{K,64}(\Phi).$$

Thus defines a continuous distribution. Its finite radial truncations converge strongly, with tail at most $C_{K,m} R^{-1} p_{K,64}(\Phi)$ above radius R.

Theorem (Strong finite part with quantitative remainder). *For every admissible fixed mollifier ρ and every $\mu > 0$,*

$$\operatorname{boxed} u_\mu^\rho = U_m + (\kappa_\rho + \log\mu)\, r$$

is the strong distributional limit

$$T_\varepsilon^\rho - \log\left(\frac{1}{\mu\varepsilon}\right) r \;\to\; u_\mu^\rho \qquad \text{in}\, \mathscr{D}'(X).$$

More precisely,

$$\left|\left\langle T_\varepsilon^\rho - \log\left(\tfrac{1}{\mu\varepsilon}\right) r - u_\mu^\rho, \Phi\right\rangle\right| \le C_{K,\rho,m}\,\varepsilon(1 + |\log\varepsilon|) p_{K,64}(\Phi), \qquad 0 < \varepsilon \le \tfrac{1}{2}.$$

The estimate includes compacts meeting $\mathrm{Rel} \cap \Delta_{13}$.

Proof. The ultraviolet decomposition gives

$$T_{\varepsilon}^{\rho} = S + \int_{1}^{\infty} \theta(\varepsilon r)^{4}(A(r) - r)\frac{\mathrm{d}r}{r}$$

$$+ \left(\int_{1}^{\infty} \theta(\varepsilon r)^{4}\frac{\mathrm{d}r}{r}\right) r + E_{\varepsilon}^{\rho}.$$

Changing variables in the scalar radial integral gives

$$\begin{aligned}\int_{1}^{\infty} \theta(\varepsilon r)^{4}\frac{\mathrm{d}r}{r} &= \log(\varepsilon^{-1}) + \kappa_{\rho} - \int_{0}^{\varepsilon}\frac{\theta(u)^{4} - 1}{u}\,\mathrm{d}u \\ &= \log(\varepsilon^{-1}) + \kappa_{\rho} + O_{\rho}(\varepsilon^{2}).\end{aligned}$$

Furthermore,

$$|\theta(\varepsilon r)^{4} - 1| \le C_{\rho}\min\{(\varepsilon r)^{2}, 1\}.$$

Lemma A therefore bounds the difference between the first radial integral in and the integral in by

$$C_{K,\rho,m}p_{K,64}(\Phi)\int_{1}^{\infty}\frac{\min\{(\varepsilon r)^{2}, 1\}}{r^{2}}\,\mathrm{d}r \le C_{K,\rho,m}^{'}\varepsilon p_{K,64}(\Phi).$$

Together with and , this proves –. The compact-test estimate implies uniform convergence on every bounded subset of $\mathscr{D}(X)$.

Shell and within-shell bounds

Let $\varepsilon_{n} = 2^{-n}$, $n \ge 1$, and define

$$E_{n}^{\rho} = T_{\varepsilon_{n+1}}^{\rho} - T_{\varepsilon_{n}}^{\rho} - (\log 2)r,$$

$$J_{n,\eta}^{\rho} = T_{2^{-(n+\eta)}}^{\rho} - T_{2^{-n}}^{\rho} - \eta(\log 2)r, \qquad 0 \le \eta \le 1.$$

The positive-cutoff integrals are continuous in the cutoff, locally uniformly on bounded sets of tests, so $J_{n,\eta}^{\rho}$ is continuous in η.

Corollary (Summable shell certificate). *For every compact $K \subset X$,*

$$|\langle E_{n}^{\rho}, \Phi\rangle| \le a_{n,K}p_{K,64}(\Phi),$$

$$\sup_{0 \le \eta \le 1} |\langle J^{\rho}_{n,\eta}, \Phi \rangle| \le b_{n,K} p_{K,64}(\Phi),$$

where one may take

$$a_{n,K} = b_{n,K} = C_{K,\rho,m}(n+1)2^{-n}.$$

In particular, $\sum_n a_{n,K} < \infty$ *and* $b_{n,K} \to 0$.

Proof. Apply with $\mu = 1$ to the two terms in each difference. For the within-shell estimate, $2^{-(n+\eta)} \in [2^{-n-1}, 2^{-n}]$, so the bound is uniform in η.

Regional certificate. Localizing by any fixed smooth partition from Section preserves the order 64. On each of its four observation regions, the shell and within-shell bounds are therefore $C_{K,\rho,m}(n+1)2^{-n}$. The constants may depend on the localization but not on n. For intersection neighborhoods they require no condition $|t - t'| \ge \delta > 0$.

Complete conversion and the real counterterm class

Let ρ_a, ρ_b be two admissible fixed mollifiers. Each is used on both slots of both insertions, as prescribed on the convention sheet.

Theorem (Complete regulator conversion). *The difference has the strong limit*

$$\text{boxed} \quad \begin{aligned} L_{ab} &= \lim_{\varepsilon \text{downarrow} 0} (T^a_\varepsilon - T^b_\varepsilon) = c_{ab} r, \\ c_{ab} &= \int_0^\infty \frac{\theta_a(u)^4 - \theta_b(u)^4}{u} \mathrm{d}u \in \mathbb{R}. \end{aligned}$$

For every common scale $\mu > 0$,

$$u^a_\mu - u^b_\mu = L_{ab}.$$

This is the complete conversion distribution: no further normal derivative, nonconstant weight on Rel, *or additional term supported on* $\mathrm{Rel} \cap \Delta_{13}$ *occurs.*

Proof. The distribution U_m in is independent of the profile. Hence

$$L_{ab} = (\kappa_{\rho_a} - \kappa_{\rho_b}) r.$$

Combining the two integrals in gives the convergent integral in . It is real because both profiles are real.

Applying to both regulators proves strong convergence of their complete difference on all joint tests. This leaves no distribution on a smaller observation stratum undetected.

Perpendicular and reality checks

Choose a real compact test ψ with $\langle r, \psi\rangle = 1$. Such a test exists since r is a nonzero positive distribution. Then

$$\langle L_{ab}, \psi\rangle = c_{ab} \in \mathbb{R}, \qquad L_{ab}^{\perp} = L_{ab} - \langle L_{ab}, \psi\rangle r = 0.$$

Thus both conditions for the prescribed real class $\mathfrak{C} = \mathbb{R}r$ are checked. The weighted-pushforward representation is completely identified:

$$L_{ab} = c_{ab} c_0 \Psi_* \nu.$$

Its weight is constant, as a consequence of the full conversion limit rather than only the leading logarithmic coefficient.

Adjoint symmetry

Let $\tau(x, y, x') = (x', y, x)$, and define

$$\langle U^{\dagger}, \Phi\rangle = \overline{\langle U, \overline{\Phi \circ \tau}\rangle}.$$

The real multiplier in implies

$$\overline{D_{\varepsilon}(z)} = D_{\varepsilon}(-z).$$

Together with , this gives

$$(T_{\varepsilon}^{\rho})^{\dagger} = T_{\varepsilon}^{\rho}.$$

The residue is real and invariant under τ, by interchanging t, t' in . Strong passage to the limit gives

$$r^{\dagger} = r, \qquad (u_{\mu}^{\rho})^{\dagger} = u_{\mu}^{\rho}, \qquad L_{ab}^{\dagger} = L_{ab}.$$

These symmetries have been proved directly; they are not used as an assumption replacing the reality calculation.

Reference profile, dilation, and cocycle

Fix one admissible reference mollifier ρ_*. Then

$$u_\mu^\rho = u_\mu^{\rho_*} + \left(\int_0^\infty \frac{\theta_\rho(u)^4 - \theta_*(u)^4}{u} \mathrm{d}u \right) r.$$

In particular,

$$L_{ab} + L_{bc} = L_{ac}, \qquad L_{aa} = 0, \qquad L_{ba} = -L_{ab}.$$

If

$$\rho_b(z) = \lambda^{-4} \rho_a(z/\lambda), \qquad \lambda > 0,$$

then the original insertion regularizations give the exact finite-cutoff identity

$$T_\varepsilon^b = T_{\lambda\varepsilon}^a, \qquad \theta_b(u) = \theta_a(\lambda u).$$

Changing variables in yields

$$\kappa_{\rho_b} = \kappa_{\rho_a} - \log\lambda, \qquad u_\mu^a - u_\mu^b = (\log\lambda) r.$$

The subtraction-scale dependence is likewise

$$u_{\mu_2}^\rho - u_{\mu_1}^\rho = \log(\mu_2/\mu_1)\, r.$$

Closed numerical statement

Theorem (Finite part for the prescribed VII-C family). *Fix the convention-sheet data, a mass $m \geq 0$, and the class of real, nonnegative, even, normalized, Euclidean-radial compact mollifiers. Use one fixed mollifier on both slots of each insertion, and keep the middle observable unsmeared.*

Then:

1. *For each $\varepsilon > 0$, the numerical family is the smooth distribution , with the prescribed product-test pairings.*

2. *The leading reflected coefficient, including its density, is exactly*

$$r = \frac{1}{64(2\pi)^3}\Psi_* v.$$

It is obtained from the exact branch integral by the normalization ledger, not imposed on that integral.

3. *For every $\mu > 0$, the prescribed subtracted family has a strong limit in $\mathscr{D}'(M_+^3)$, with the fixed-order estimate . This includes the full reflected-diagonal intersection.*

4. *The regulator-independent ultraviolet decomposition is . Its radial and regulator-placement errors satisfy*

$$\begin{aligned} \langle A(r) - r, \Phi\rangle &= O_{K,m}(r^{-1})p_{K,64}(\Phi), \\ \langle E_\varepsilon^\rho, \Phi\rangle &= O_{K,\rho,m}(\varepsilon(1+|\log\varepsilon|))p_{K,64}(\Phi). \end{aligned}$$

5. *For any two admissible mollifiers, the complete conversion is . In particular, $L_{ab}^{\perp} = 0$ and $c_{ab} \in \mathbb{R}$. The fixed class $\mathfrak{C} = \mathbb{R}r$ suffices. These conclusions concern only the numerical prescription defined in this paper.*

Proof. Proposition proves (i). Proposition proves (ii) as an equality on arbitrary joint tests. Lemma , Lemma A, and Lemma give the ultraviolet decomposition and its uniform error estimates. Theorem proves (iii), while Section records the observation and momentum coverage. Theorem proves (v).

Book expansion notes (t433u)

The finite part constructed here and the VII-C mixed-Wick obstruction [1] coexist. The finite part is not promoted to a product or a state. Observation-boundary source record: [7].

Standing ledger. Not spacetime emergence; not Einstein dynamics; not a Page curve, island, bounce, vacuum-energy selection, or de Sitter state as an Enter theorem; not a derived metric. O.1 stays NOT_CLOSED. XVI-N is not discharged. XV-B is not opened. No quantum coefficient is admitted. Host C is not created. Dom (T) remains undetermined; J2 remains unresolved. Problems 6, 7, and 9 are named, not solved. Gate 0 and ST4b stay NOT_SUPPLIED. Host-A stays NAMED_COMPARISON_ONLY $\{GI, CB\}$. The shredder is not

Gibbons–Hawking. Cited sheets are source records, not premises. Conjectures stay conjectures. wired=false.

References

[1] N. Cheung, A mixed-Wick obstruction to a normal-topology boundary extension, in *Observation Boundaries and Finite Instruments: Series VII*, Cheung Ngo Medical Limited (2026).

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